An endogenous growth model of premature deindustrialization

Komla Avoumatsodo* Wilfried Koch† Isambert Leunga Noukwe‡

Abstract

Many recent industrializers seem to be experiencing a lower peak in manufacturing labor share, and the peak is occurring at a much lower level of development relative to what earlier industrializers experienced, Rodrik (2016) called this phenomenon premature deindustrialization (PD). Recent studies show that heterogeneity in sectoral productivity across sectors and countries is the main driving of PD. Using a Schumpeterian growth model of structural change, we show analytically how heterogeneity in productivity affects the labor share at the peak in the industry sector and the GDP at that peak through the ratio of the gap between productivity growth rates in agriculture and industry sectors and the gap between productivity growth rates in industry and services sectors. This ratio captures the tension between two opposing forces: the force which pushes workers from agriculture into industry to the force that pulls workers from industry into services. Through the lens of our endogenous growth model, we show that PD can result from cross-country heterogeneity in the initial levels of productivity and in the parameters governing sectoral innovation, ie. the efficiency of R& D activity and the size of innovation in each sector.

Keywords: Aggregate balanced growth; Premature deindustrialization; Endogenous growth; Vertical innovation; Structural change; R&D.

JEL classification: O11, O14, O31, O32, O41

*School of Economics, University of Northern British Columbia, komla.avoumatsodo@unbc.ca

†Department of Economics, Université du Quebec à Montréal : koch.wilfried@uqam.ca

‡Department of Economics, Université de Sherbrooke, isambert.leunga.noukwe@usherbrooke.ca

1 Introduction

Structural change refers to the reallocation of economic activity across the broad sectors of agriculture, manufacturing, and services. As countries grow richer, the share of agriculture, whether measured in employment or value-added shrinks, the share of services rises, and the share of the manufacturing sector exhibits a hump-shaped pattern, increasing at low levels of development (i.e., the industrialization phase), reaching a peak, and then declining in the later stages of development (i.e., the deindustrialization phase). Recent research has documented heterogeneity in the patterns of structural change across countries. Rodrik (2016) shows that more recent industrializers entered the stage of deindustrialization at lower income levels with lower peaks of manufacturing shares, compared to more advanced economies that had industrialized earlier. He called this phenomenon "premature deindustrialization" (PD).

Recent research investigating the mechanisms behind premature deindustrialization argues that this phenomenon is the result of heterogeneous productivity across sectors and across countries (Huneeus & Rogerson 2020, Fujiwara & Matsuyama 2022 and Sposi et al. 2021). This paper aims to answer two questions. First, how does the heterogeneity in productivity affects the labor share in the industry sector at its peak and the GDP at that peak? Second, what are the determinants of heterogeneity in productivity across countries?

To address these questions, we develop a multi-sector Schumpeterian growth model in which structural change is driven by sector-biased productivity growth generated by asymmetric technology of innovation across sectors. Following Ngai & Pissarides (2007) and Boppart (2014), we focus on the Baumol effect which emphasizes the importance of non-unitary sectoral substitution elasticities in conjunction with unequal productivity growth across sectors. This mechanism is the most important driver of the structural change (Dennis & İşcan 2009, Uy et al. 2013 and Święcki 2017)[1]. In our endogenous growth model, there are three consumption goods: agriculture, industry, and services, which are produced competitively using a continuum of intermediate goods. There is free entry into innovation and each firm performs R&D intending to innovate in a line of intermediate

[1] Other drivers behind structural change are the income effect which emphasizes on the income elasticities of demand for each sectoral good differ from one (e.g. Kongsamut et al. 2001, Herrendorf et al. 2013 and Uy et al. 2013) and International trade which emphasizes on change of labor share due to specialization which accompanied sectoral productivities growth (e.g. **?**, **?** and **?**).

good and becomes a monopolist producer for this line. Sectoral productivity growth results from vertical innovations through quality ladder setup.

We have four key findings. First, we show that under plausible conditions, our model is consistent with the empirically observed pattern of structural change as well as aggregate balanced growth in which structural change takes place underneath. Second, we show analytically that heterogeneity in productivity across sectors affects the labor share at the peak in the industry sector and the *GDP* at that peak through the ratio of the gap between productivity growth rates in agriculture and industry sectors and of the gap between productivity growth rates in industry and services sectors, we call this ratio the *Sectoral Productivity Growth Gap Index (SPGI)*. This index captures the tension between two opposing forces: the force that pushes labor from agriculture to industry and the force that pulls labor out of the industry for services. A large *SPGI* would imply a reallocation of workers from agriculture to the industry greater than the reallocation of workers from the industry to the services sector, while a smaller *SPGI* would imply the opposite effect. Third, through the lens of our Schumpeterian growth model, we show how innovation parameters such as the efficiency of R&D activity and the size of innovation in each sector[2] can affect the *SGPI*, and then the labor share in the industry at its peak and the *GDP* at that peak. We show how the heterogeneity of these parameters across countries and sectors can explain PD and the subjoined conditions necessary to obtain these results. Four, we show that premature deindustrialization can result from heterogeneity across countries in the level of initial productivity across sectors and countries.

Our paper is related to two strands of the structural change literature. The first strand is the research that documents the premature deindustrialization of a large sample of countries and includes Rodrik (2016), Felipe & Mehta (2016), Felipe et al. (2019) and Haraguchi et al. (2017). The second strand pertains to the literature on the endogenous growth model of structural change. This research includes Zhang (2018*a*), Bondarev & Greiner (2019) and Guilló et al. (2011) who do not examine whether structural change is consistent with aggregate balanced growth and on the other hand Boppart & Weiss (2013), Herrendorf & Valentinyi (2015) and Hori et al. (2018) who build endogenous growth models consistent with aggregate balanced growth. All previous papers do not examine the mechanism behind premature deindustrialization.

[2]The size of innovation refers to the magnitude of enhancement of productivity resulting in each innovation.

Our paper is also close to the growing literature developing models of premature deindustrialization including Huneeus & Rogerson (2020), Fujiwara & Matsuyama (2022) and Sposi et al. (2021). Like us, these three papers develop models of deindustrialization which emphasize sectoral productivity growth. Huneeus & Rogerson (2020) show that heterogeneous patterns of catch-up in sectoral productivities across countries are key drivers of both structural change and deindustrialization. Fujiwara & Matsuyama (2022) show that heterogeneity in technology gaps between sectors and across countries can explain the declining "hump" pattern for the later industrializers, as well as the lower per capita income at that hump. Sposi et al. (2021) use a Ricardian setting to investigate the role of trade integration and sector-biased productivity growth on deindustrialization. They find that sector-biased productivity growth alone can explain about 60 percent of patterns of deindustrialization. They also find that the rapid fall of trade costs in the manufacturing sector has contributed to deindustrialization.

Our paper supports the finding of these papers which argue that heterogeneity in sectoral productivity across sectors and countries can explain deindustrialization. We contribute to this literature by showing how heterogeneity in sectoral productivity affects labor share in industry at its peak and the value of GDP at that peak. Moreover, while all these papers consider exogenous productivity growth, we use an endogenous growth model which allows us to show that heterogeneity in sectoral's innovation can explain premature deindustrialization.

The paper is organized as follows. Section 3.2 presents our model, Section 3.3 analyzes the aggregate balanced growth path. In Section 3.4, we analyze the deindustrialization along the aggregate balanced growth path while the final Section concludes.

2 Model

In this section, we describe our endogenous growth model of structural change in the Schumpeterian framework. There are three consumption goods in the economy: agriculture, industry/manufacturing, and services denoted by $j \in a, m, s$ respectively. The final good in each sector is produced competitively using a continuum of intermediate goods. Each variety of intermediate goods is produced by monopolistic firms using labor. Labor is fully mobile across sectors. There is free entry in R&D activity and the monopolist firms operating in each sector use the final goods in that sector for R&D. The economy is closed, and the time is continuous and denoted by $t \in [0, \infty)$.

2.1 Households

The economy is populated by a representative household with constant relative risk aversion (CRRA) preferences given by

$$U = \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} dt, \quad \theta > 0,\ \theta \neq 1, \tag{1}$$

where ρ is the discount factor, θ stands for the inverse of the elasticity of substitution, and $C(t)$ denotes an aggregator function for the consumption of final goods at period t. We assume the homothetic constant elasticity of substitution (CES) form

$$C(t) = \left[\sum_{j=a,m,s} \eta_j^{\frac{1}{\sigma}} C_j(t)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \qquad \sum_{j=a,m,s} \eta_j = 1, \tag{2}$$

where η_k sum to one and represent the weights assigned to the consumption of the final good of each sector, $C_j(t)$ is the consumption of the sector j final good at time t and σ is the elasticity of substitution between sectoral goods, we assume that sectoral goods are gross complements, so that $0 < \sigma < 1$.

The representative household inelastically supplies L units of labor in each period. The total income received by the representative consumer at time t is the sum of labor income, $w(t)L$, and asset income $r(t)\mathcal{A}(t)$, where $w(t)$ is the wage rate at time t, $r(t)$ is the real interest rate at time t, and $\mathcal{A}(t)$ is the representative household assets at time t.

The representative household maximizes its utility defined in (1) and (2) subject to the flow budget constraint

$$\dot{\mathcal{A}}(t) + E(t) \leq w(t)L + r(t)\mathcal{A}(t), \tag{3}$$

and the usual no-Ponzi condition,

$$\lim_{t \to \infty} \left[e^{-\int_0^t r(u)du} \mathcal{A}(t) \right] \geq 0.$$

$E(t)$ is the household total expenditure at date t,

$$E(t) = \sum_{j=a,m,s} P_j(t)C_j(t),$$

where $P_j(t)$ is the price of sector j final good at date t. Throughout, we normalize the aggregate price index at any date to one:

$$P(t) = \left[\sum_{j=a,m,s} \eta_j P_j(t)^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \equiv 1. \tag{4}$$

The representative household's intertemporal optimization problem delivers the standard Euler equation,

$$\frac{\dot{E}(t)}{E(t)} = \frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho}{\theta}. \tag{5}$$

This is the familiar form of the Euler equation which is consistent with a constant growth path along which the interest rate is constant.

The first-order conditions of the household optimization problem imply that

$$C_j(t) = \eta_j \kappa(t)^{-\frac{1}{\theta}} P_j(t)^{-\sigma}, \tag{6}$$

where $\kappa(t)$ is the costate variable associated with the representative consumer's intertemporal budget constraint.

Differentiating equation (6) with respect to time t implies that the evolution of sector j final consumption $C_j(t)$ follows

$$\frac{\dot{C}_j(t)}{C_j(t)} = \frac{r(t)-\rho}{\theta} - \sigma \frac{\dot{P}_j(t)}{P_j(t)} \quad \forall j \in \{a,\, m,\, s\}. \tag{7}$$

See Appendix A.2 for proof.

2.2 Technology

Turning to the production side, there is one final good in each sector which is produced with intermediate goods. Each intermediate good producer uses labor as a production factor. In this section, we present the production technology of both final and intermediate goods.

2.2.1 Final goods

There is a unique final good by sector that is produced using a continuum of intermediate goods as inputs. The production function for the single final good in sector j is

$$Y_j(t) = \left[\int_0^1 A(\nu_j, t)^{\frac{1}{\varepsilon}} x(\nu_j, t)^{\frac{\varepsilon-1}{\varepsilon}} d\nu_j\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{8}$$

where $x(\nu_j, t)$ is the flow of intermediate good ν_j used in the production of sector j final good at time t, $A(\nu_j, t)$ denote the quality or productivity of ν_j intermediate good and $\varepsilon > 1$ is the elasticity of substitution between varieties of intermediate goods. We assume that intermediate goods fully depreciate after use.

The sectoral final goods are produced competitively. So, faced with the given price of the intermediate good ν_j, which is denoted by $p(\nu_j, t)$, the profit maximization problem of the final good producer in sector j is given by

$$\max_{x(\nu_j,t)} P_j(t) \left[\int_0^1 A(\nu_j, t)^{\frac{1}{\varepsilon}} x(\nu_j, t)^{\frac{\varepsilon-1}{\varepsilon}} d\nu_j\right]^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 p(\nu_j, t) x(\nu_j, t) d\nu_j.$$

The first order condition of the above profit maximization problem leads to the inverse demand function for intermediate good ν_j is given by

$$p(\nu_j, t) = P_j(t) \left(\frac{Y_j(t)}{x(\nu_j, t)}\right)^{\frac{1}{\varepsilon}} A(\nu_j, t)^{\frac{1}{\varepsilon}}. \tag{9}$$

2.2.2 Intermediate good production

Intermediate good ν_j is produced by the monopolist who has the best (leading-edge) technology $A(\nu_j, t)$ in that product line at the date t. At any given point in time, each leading firm has access to a technology capable of producing one unit of intermediate variety ν_j with ϕ_j units of labor.

The monopolistic producer selects its price to maximize profits

$$\begin{aligned} \pi(\nu_j, t) &= p(\nu_j, t) x(\nu_j, t) - w(t)\phi_j x(\nu_j, t) \qquad (10) \\ \text{s.t} \qquad & p(\nu_j, t) = P_j(t) \left(\frac{Y_j(t)}{x(\nu_j, t)}\right)^{\frac{1}{\varepsilon}} A(\nu_j, t)^{\frac{1}{\varepsilon}}. \end{aligned}$$

The price and the production level of intermediate good ν_j follow from this maximization as

$$p(\nu_j,t)=\frac{\varepsilon}{\varepsilon-1}\phi_j w(t) \quad \text{and} \quad x(\nu_j,t)=\left(\frac{\varepsilon-1}{\varepsilon}\frac{P_j(t)}{\phi_j w(t)}\right)^{\varepsilon} Y_j(t)A(\nu_j,t). \tag{11}$$

2.2.3 Sectoral and aggregate outputs

The price of the final good in sector j equals

$$P_j(t)=\left(\int_0^1 A(\nu_j,t)p(\nu_j,t)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}=\frac{\varepsilon}{\varepsilon-1}\phi_j w(t)A_j(t)^{-\frac{1}{\varepsilon-1}}. \tag{12}$$

where $A_j(t)$ is the average of all intermediate goods productivities in sector j at time t defined as follows

$$A_j(t)\equiv\int_0^1 A(\nu_j,t)d\nu_j. \tag{13}$$

By substituting the price of sectoral final goods in (12) in the expression of the normalized price index in (4), we get

$$w(t)=\frac{\varepsilon-1}{\varepsilon}A(t), \tag{14}$$

where

$$A(t)\equiv\left[\sum_{j=a,m,s}\eta_j\left(\phi_j A_j(t)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma}\right]^{\frac{-1}{1-\sigma}}. \tag{15}$$

and refer to the aggregate productivity in our economy. Thus, the real wage is proportional to aggregate productivity.

Substituting the price of sectoral final goods in (12) into equation (11) we obtain also

$$x(\nu_j,t)=A_j(t)^{-\frac{\varepsilon}{\varepsilon-1}}Y_j(t)A(\nu_j,t). \tag{16}$$

Recall that the production of each unit of the intermediate goods ν_j requires ϕ_j units of labor. Thus, the number of units of labor $L_j(t)$ use to produce all intermediate goods in sector j at time t is

$$L_j(t) \quad = \quad \int_0^1 \phi_j x(\nu_j,t)d\nu_j. \tag{17}$$

Combining equations (16) and (17) leads to the expression of the production function of sector j final good

$$Y_j(t) = \phi_j^{-1} A_j(t)^{\frac{1}{\varepsilon-1}} L_j(t). \tag{18}$$

The equilibrium profits can then be computed as

$$\pi(\nu_j, t) = \frac{1}{\varepsilon - 1} w(t) x(\nu_j, t) = \frac{1}{\varepsilon} A(t) L_j(t) \frac{A(\nu_j, t)}{A_j(t)}. \tag{19}$$

This last equation shows that profits grow with aggregate productivity and the distance of the productivity of variety ν_j from the average productivity in sector j.

Total output at period t is given by

$$Y(t) \equiv \sum_{j=a,m,s} P_j(t) Y_j(t) = A(t) L(t), \tag{20}$$

and the total profit generated by the monopolistic producers at period t is

$$\Pi(t) = \sum_{j=a,m,s} \int_0^1 \pi(\nu_j, t) d\nu_j = \frac{1}{\varepsilon} A(t) L(t). \tag{21}$$

We denote by $GDP(t)$ the real GDP in period t. In our model, $GDP(t)$ is equal to the sum of valued-added generated in all sectors.

$$\begin{aligned} GDP(t) &= \sum_{j=a,m,s} \left[\left(P_j(t) Y_j(t) - \int_0^1 p(\nu_j, t) x(\nu_j, t) d\nu_j \right) + \int_0^1 p(\nu_j, t) x(\nu_j, t) d\nu_j \right] \\ &= Y(t). \end{aligned}$$

Since at labor market equilibrium $L(t) = L$, the equilibrium real GDP per worker and the aggregate profits in the economy are proportional to the aggregate productivity. Thus, the growth of real GDP per worker in the model is determined by the growth of the aggregate productivity in the economy. We will now analyze the mechanism and motivations for innovation before closing the model presentation.

2.2.4 Innovation, productivity growth, and entry

There is an infinite number of innovators that can freely conduct the R&D. The innovators compete to discover the next generation of machines. We assume that each innovation at date t in any variety ν_j allows the innovator to produce this variety with "leading-edge" technology. The previous incumbent in variety ν_j, whose technology is no longer on the leading edge, will be displaced. We consider the quality/productivity ladder setup. Innovation on each line of intermediate good ν_j increases the productivity of this variety by a constant $\gamma_j > 1$, which we call the size of innovation in the sector j. Therefore, if an innovation occurs with probability $\mu(\nu_j, t)$ in intermediate good ν_j, then productivity increases from $A_j(\nu_j, t)$ to $\gamma_j A_j(\nu_j, t)$. If, on the contrary, there is no innovation in ν_j at date t (with probability $1 - \mu(\nu_j, t)$), the level of productivity of the variety ν_j remains at $A_j(\nu_j, t)$. We can write

$$A(\nu_j, t + \Delta t) = \begin{cases} \gamma_j A(\nu_j, t) & \text{with probability} \quad \mu(\nu_j, t)\Delta t + o(\Delta t) \\ A(\nu_j, t) & \text{with probability} \quad (1 - \mu(\nu_j, t)\Delta t) + o(\Delta t) \end{cases}$$

The expected level of productivity of variety ν_j at time $t + \Delta t$ is given by

$$A(\nu_j, t + \Delta t) = \mu(\nu_j, t)\gamma_j A(\nu_j, t)\Delta t + A(\nu_j, t)\left[1 - \mu(\nu_j, t)\Delta t\right] + o(\Delta t).$$

Aggregating over ν_j, and taking the limit where Δt is close to zero, we derive the growth rate of aggregate productivity in the sector j as follows

$$\dot{A}_j(t) = (\gamma_j - 1)\int_0^1 \mu(\nu_j, t)A(\nu_j, t)d\nu_j. \tag{22}$$

Equation (22) shows that productivity growth rate in sector j depends positively on the size of innovation in that sector γ_j.

We now turn to the innovation technology. We assume that each innovator in sector j uses the final good of that sector to perform R&D. The probability $\mu(\nu_j, t)$ of innovation in variety ν_j of sector j is given by

$$\mu(\nu_j, t) = \lambda_j Z(\nu_j, t)A_j(t)^{\psi}A(t)^{\zeta}, \tag{23}$$

where $Z(\nu_j, t)$ is the number of units of sector j final good spent in R&D on intermediate good ν_j, $A_j(t)$ is the average productivity of all intermediate goods in sector j at time t defined in equation (13) and $A(t)$ is the aggregate productivity in the economy defined in equation (30). The parameters ψ and ζ measure the degree of spillover effects of the current levels of sector j and aggregate productivities to future technology invention, respectively. As in Zhang (2018*a*), we assume $\zeta,\, \psi \in (-\infty, 0)$ to capture the "fishing out" theory in which the rate of innovation decreases with the level of current technology. Finally, the parameter $\lambda_j > 0$ refers to the efficiency of R&D activity in sector j.

Let $V(\nu_j, t)$ be the value function of the producer of the intermediate good ν_j. The objective of a potential entrant is to choose $Z(\nu_j, t)$ at each period to maximize the flow of expected profits from the research by solving the following optimization problem

$$\begin{aligned} &\max_{Z(\nu_j,t)}\ \mu_j(\nu_j,t)\, V(\nu_j,t) - P_j(t)\, Z(\nu_j,t), \\ &\text{s.t} \quad \mu(\nu_j,t) = \lambda_j Z(\nu_j,t) A_j(t)^{\psi} A(t)^{\zeta}. \end{aligned}$$

Free entry implies that the present value of the monopoly profits for the higher quality of the variety ν_j equals the entry cost into the production market of this variety

$$V(\nu_j,t) = \frac{P_j(t)}{\lambda_j A_j(t)^{\psi} A(t)^{\zeta}} \tag{24}$$

Moreover, innovator in variety ν_j realizes a stream of future profits with a present value of

$$V(\nu_j,t) = \int_t^{+\infty} e^{-\int_t^s [r(u)+\mu_j(\nu_j,u)]du}\, \pi(\nu_j,s) ds. \tag{25}$$

Differentiating equation (25) with respect to time t, we obtain the following Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\dot{V}(\nu_j,t)}{V(\nu_j,t)} = r(t) + \mu(\nu_j,t) - \frac{\pi(\nu_j,t)}{V(\nu_j,t)}, \tag{26}$$

Substituting equations (19), (23) and (24) in relation (26) and aggregating with respect to ν_j yields

$$\frac{\dot{w}(t)}{w(t)} - \left(\frac{1}{\varepsilon-1} + \psi\right)\frac{\dot{A}_j(t)}{A_j(t)} - \zeta\frac{\dot{A}(t)}{A(t)} = r(t) + \left(Z_j(t) - \frac{\gamma_j}{\varepsilon\phi_j} L_j(t) A_j(t)^{\frac{1}{\varepsilon-1}}\right)\lambda_j A_j(t)^{\psi} A(t)^{\zeta}. \tag{27}$$

where $Z_j(t) = \displaystyle\int_0^1 Z(\nu_j,t)d\nu_j$ is the total unit of sector j final good use in R&D activity in that sector.

2.3 Market Clearing and Dynamic Equilibrium

In this subsection, we describe the market clearing conditions.

Goods market: Sectoral final goods are used for household consumption and R&D expenditure. Then,

$$C_j(t) + Z_j(t) = Y_j(t) \qquad \forall j \in \{a, m, s\}\,. \tag{28}$$

Labor market: The labor market clearing conditions can be written as

$$L_a(t) + L_m(t) + L_s(t) = L(t), \quad \text{and} \quad L(t) = L. \tag{29}$$

Asset market: Finally, asset market clearing implies

$$\mathcal{A}(t) = \sum_{j=a,m,s} \int_0^1 V(\nu_j, t) d\nu_j = \sum_{j=a,m,s} \frac{P_j(t)}{\lambda_j A_j(t)^{\psi} A(t)^{\varsigma}}. \tag{30}$$

Dynamic equilibrium: Dynamic equilibrium in this economy consists of a collection of time paths of

- consumption levels $[C_a(t),\, C_m(t),\, C_s(t)]_{t=0}^{\infty}$,
- number of workers $[L_a(t),\, L_m(t),\, L_s(t)]_{t=0}^{\infty}$,
- qualities leading-edge varieties $\left[\{A(\nu_j, t)\}_{\nu_j=0}^{1},\ j = a, m, s\right]_{t=0}^{\infty}$,
- demand of intermediate goods $\left[\{x(\nu_j, t),\}_{\nu_j=0}^{1},\ j = a, m, s\right]_{t=0}^{\infty}$,
- R&D expenditures $\left[\{\ Z(\nu_j, t)\}_{\nu_j=0}^{1},\ j = a, m, s\right]_{t=0}^{\infty}$,
- wage rates, interest rates, prices of final goods $[w(t),\, r(t), P_j(t),\ j = a, m, s]_{t=0}^{\infty}$ and prices of intermediate goods $\left[\{p(\nu_j, t)\}_{\nu_j=0}^{1},\ j = a, m, s\right]_{t=0}^{\infty}$,

such that given wage rates, interest rates, and prices of final and intermediate goods, the representative household maximizes its utility, competitive final goods producers choose quantities to maximize profits, intermediate goods monopolists set prices to maximize profits, varieties productivities evolve according to the innovation technology given the R&D expenditure, the R&D expenditure on each variety is determined by free entry and all markets clear.

3 Balanced growth and desindustrialization

This section examines whether our model is consistent with the empirically observed patterns of structural change, in particular hump shape in the industry sector and aggregate balanced growth.

3.1 Balanced growth

We follow Ngai & Pissarides (2007) and Herrendorf et al. (2018) and define an aggregate balanced growth path (ABGP henceforth) in our economy as follows.

Definition 1 ***(Balanced Growth)*** *The aggregate balanced growth path (ABGP) is defined as an equilibrium growth path where aggregate consumption and output grow at the same constant rate.*

Note that this definition which requires balanced growth for aggregate variables does not require balanced growth for sectoral variables (equal growth rates across sector). Hence, it allows for structural change along the ABGP.

Lemma 1 *Along ABGP*

1. *the interest rate $r(t)$ is constant*

2. *all aggregate variables grow at the same rate:*

$$\gamma \equiv \frac{\dot{E}(t)}{E(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{\mathcal{A}}(t)}{\mathcal{A}(t)} = \frac{\dot{C}(t)}{C(t)} = \frac{\dot{w}(t)}{w(t)} = \frac{\dot{A}(t)}{A(t)}. \tag{31}$$

Proof. Recall the Euler equation defined in relation (5)

$$\frac{\dot{E}(t)}{E(t)} = \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\theta}. \tag{32}$$

This equation implies that $E(t)$ and $C(t)$ grow at the same rate and $r(t)$ is constant along ABGP. We will note it from now on by r. Thus,

$$\gamma = \frac{r - \rho}{\theta}.$$

We assume that the model's parameters are such that this growth rate is positive.

Moreover, the budget constraint (3) can be rewritten as

$$\frac{\dot{\mathcal{A}}(t)}{\mathcal{A}(t)} + \frac{E(t)}{\mathcal{A}(t)} = \frac{w(t)L}{\mathcal{A}(t)} + r(t).$$

This implies that the representative household assets $\mathcal{A}(t)$, the total expenditure $E(t)$, and the real wage $w(t)$ should grow at the same constant rate along ABGP. Moreover, recall that

$$w(t) = \frac{\varepsilon - 1}{\varepsilon} A(t) \quad \text{and} \quad Y(t) = A(t)L.$$

Differentiating these relations with respect to time t leads to,

$$\frac{\dot{w}(t)}{w(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{A}(t)}{A(t)},$$

which combined with the previous equation shows that

$$\frac{\dot{\mathcal{A}}(t)}{\mathcal{A}(t)} = \frac{E(t)}{\mathcal{A}(t)} = \frac{\dot{w}(t)}{w(t)} = \frac{\dot{Y}(t)}{Y(t)} = \frac{\dot{A}(t)}{A(t)}. \blacksquare$$

The question that arises here is whether there is structural change along the ABGP. We address this question in the next lemma. The following assumption imposes restrictions on technology and preferences parameters to ensure a constant growth rate in sectoral consumption and output along the ABGP. It is worth noting that this type of restriction is common in Schumpeterian endogenous growth literature. Noting also that the growth rate of consumption is different across sectors as well as the growth rate of output.

Assumption 1 *(i)* $\zeta = -(1-\sigma)$, *(ii)* $\psi = -\dfrac{\sigma}{\varepsilon - 1}$.

Lemma 2 *If Assumption 3.1 holds, structural change takes place along the ABGP. That is, the employment shares of sectors agriculture, industry, and services change over time along the ABGP.*

$$\frac{\dot{L}_j(t)}{L_j(t)} = (1-\sigma)\gamma - \frac{1-\sigma}{\varepsilon - 1}\frac{\dot{A}_j(t)}{A_j(t)} \tag{33}$$

$$\frac{\dot{Y}_j(t)}{Y_j(t)} = \frac{\dot{C}_j(t)}{C_j(t)} = (1-\sigma)\gamma + \frac{\sigma}{\varepsilon - 1}\frac{\dot{A}_j(t)}{A_j(t)}. \tag{34}$$

Proof. Differentiating the price in equation (12) with respect to t gives

$$\frac{\dot{P}_j(t)}{P_j(t)} = \frac{\dot{w}(t)}{w(t)} - \frac{1}{\varepsilon - 1}\frac{\dot{A}_j(t)}{A_j(t)},$$

which we use for substitution into Euler equation of $C_j(t)$ in equation (7) and obtain

$$\frac{\dot{C}_j(t)}{C_j(t)} = \frac{r-\rho}{\theta} - \sigma\frac{\dot{w}(t)}{w(t)} + \frac{\sigma}{\varepsilon - 1}\frac{\dot{A}_j(t)}{A_j(t)} = (1-\sigma)\gamma + \frac{\sigma}{\varepsilon - 1}\frac{\dot{A}_j(t)}{A_j(t)}. \tag{35}$$

The resources constraint in sector j is given by

$$C_j(t) + Z_j(t) = Y_j(t) = \phi_j^{-1} A_j(t)^{\frac{1}{\varepsilon-1}} L_j(t).$$

This equation tells us that along the ABGP, consumption, R&D expenditures and output in sector j grow at the same rate.

$$\frac{\dot{C}_j(t)}{C_j(t)} = \frac{\dot{Z}_j(t)}{Z_j(t)} = \frac{\dot{Y}_j(t)}{Y_j(t)} = \frac{1}{\varepsilon - 1}\frac{\dot{A}_j(t)}{A_j(t)} + \frac{\dot{L}_j(t)}{L_j(t)}. \tag{36}$$

Thus, along the ABGP, consumption, R&D expenditures are proportional to output in sector j. We denote by z_j and c_j the corresponding proportionality coefficients. We can write

$$Z_j(t) = z_j Y_j(t) = z_j \phi_j^{-1} L_j(t) A_j(t)^{\frac{1}{\varepsilon-1}}, \tag{37}$$

$$C_j(t) = c_j Y_j(t) = c_j \phi_j^{-1} L_j(t) A_j(t)^{\frac{1}{\varepsilon-1}}. \tag{38}$$

Furthermore, the free entry condition of R&D in equation (27) can be rewritten as

$$(1-\zeta)\frac{\dot{A}(t)}{A(t)} - \left(\psi + \frac{1}{\varepsilon - 1}\right)\frac{\dot{A}_j(t)}{A_j(t)} = r(t) + \left(z_j - \frac{\gamma_j}{\varepsilon}\right) Y_j(t)\lambda_j A_j(t)^{\psi} A(t)^{\zeta}. \tag{39}$$

This equation implies that $Y_j(t)A_j(t)^{\psi}A(t)^{\zeta}$ has to be constant along the ABGP. Knowing this, we deduce that

$$\frac{\dot{Y}_j(t)}{Y_j(t)} = -\psi\frac{\dot{A}_j(t)}{A_j(t)} - \zeta\frac{\dot{A}(t)}{A(t)} = -\psi\frac{\dot{A}_j(t)}{A_j(t)} - \zeta\gamma. \tag{40}$$

Putting equations (35) and (40) together yields the following restriction on parameters

$$\zeta = -(1-\sigma) \quad \text{and} \quad \psi = -\frac{\sigma}{\varepsilon - 1}. \tag{41}$$

Given these restrictions, (35) implies that the growth rate of consumption and output in sector j is

$$\frac{\dot{Y}_j(t)}{Y_j(t)} = \frac{\dot{C}_j(t)}{C_j(t)} = (1-\sigma)\gamma + \frac{\sigma}{\varepsilon - 1}\frac{\dot{A}_j(t)}{A_j(t},$$

which together with (36) yields

$$\frac{\dot{L}_j(t)}{L_j(t)} = (1-\sigma)\gamma - \frac{1-\sigma}{\varepsilon - 1}\frac{\dot{A}_j(t)}{A_j(t)}.\blacksquare$$

Lemma 3.2 shows that the labor growth rate in a given sector is negatively related to productivity growth in that sector, while the output growth in a sector is positively related to the productivity growth in that sector. This lemma also shows that along the ABGP, the labor reallocation across the agricultural, industrial, and services sectors is determined by asymmetric productivity growth across sectors. Since $\varepsilon > 1$ and $0 < \sigma < 1$, workers will move from the sector with high productivity growth to sectors with low productivity growth. The next proposition gives the expression of productivity growth rate and sectoral labor in each sector along ABGP.

Proposition 1 *Along the ABGP, sectoral productivities grow at the constant rates given by*

$$\frac{\dot{A}_j(t)}{A_j(t)} = \frac{\gamma_j - 1}{(\gamma_j - 1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}}\left[\frac{\lambda_j\eta_j}{\phi_j^\sigma\varepsilon}\left(\frac{\varepsilon-1}{\varepsilon}L + (\gamma - r)\frac{\phi_{j_o}^\sigma}{\lambda_{j_o}\eta_{j_o}}\right)\gamma_j + \Gamma\right]. \tag{42}$$

The expression for labor in sector j is given by

$$\begin{aligned} L_j(t) &= \left[\eta_j\phi_j^{1-\sigma}\left((\gamma_j - 1)\frac{1-\sigma}{\varepsilon - 1} + 1\right)\left(\frac{\varepsilon - 1}{\varepsilon}L + (\gamma - r)\frac{\phi_{j_o}^\sigma}{\lambda_{j_o}\eta_{j_o}}\right) + \frac{\phi_j}{\lambda_j}\Gamma\right] \\ &\qquad \times A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}}A(t)^{1-\sigma}\left((\gamma_j - 1)\frac{1-\sigma}{\varepsilon - 1} + 1 - \frac{\gamma_j}{\varepsilon}\right)^{-1} \end{aligned} \tag{43}$$

where

$$\Gamma \equiv (2 - \sigma - \theta)\gamma - \rho, \quad \textit{and} \quad j_o = \arg\min_j\left\{\frac{\dot{A}_j(t)}{A_j(t)},\ j = a, m, s\right\}.$$

Proof. See Appendix A.2.

Let denote g_j the growth rate of productivity $A_j(t)$ along the ABGP according to Proposition 3.1.

It remains to characterize the labor share along the ABGP. Equation (34) implies that workers shifts over time from the high productivity growth to the low productivity growth sector. Thus, we focus our attention on cases where $g_a > g_m > g_s$ to allow our model prediction to be consistent with the empirically observed patterns of structural change characterized by the decreasing labor share in agriculture, the hump-shape in industry, and the increase in the labor share in services. As documented by several authors including Herrendorf et al. (2014), this ranking of sectoral productivity growth rates is verified empirically for a large sample of countries. Therefore, this is consistent with the following assumption.

Assumption 2 *We assume that model parameters are such that*

$$g_a > g_m > g_s,$$

where g_j refers to the productivity growth rate in sector j.

Lemma 3 *If Assumption 3.2 holds, along the ABGP*

1. *the labor share in agriculture monotonously decreases over time , the labor share in services monotonously increases over time,*
2. *the labor share in industry exhibits a "hump-shaped" pattern and the date of the peak is given by*

$$t^* = \frac{\varepsilon - 1}{1 - \sigma} \frac{1}{g_a - g_s} \log \left[\frac{L_a(0)}{L_s(0)} \frac{g_a - g_m}{g_m - g_s} \right], \tag{44}$$

where the expression for $L_j(0)$ is given by equation (43), which depends only on the initial sectoral productivities and the model's parameters.

Fujiwara & Matsuyama (2022) find similar results of the date for peak in industry. However, they focus instead on how sectoral technology adoption lags affect this date and the labor share in the industry and GDP at the peak.

Proof. Let $s_j(t)$ be the labor share in sector j i.e.

$$s_j(t) = \frac{L_j(t)}{\sum\limits_{k=a,m,s} L_k(t)}.$$

According to equation (33), labor in sector j grows at a constant rate and can be written as

$$L_j(t) = L_j(0) \exp\left[\left((1-\sigma)\gamma - \frac{1-\sigma}{\varepsilon-1} g_j\right) t\right],$$

where

$$\begin{aligned} L_j(0) \quad &= \quad \left[\eta_j \phi_j^{1-\sigma}\left((\gamma_j - 1)\frac{1-\sigma}{\varepsilon-1} + 1\right)\left(\frac{\varepsilon-1}{\varepsilon}L - (\gamma - r)\frac{\phi_s^\sigma}{\lambda_s \eta_s}\right) + \frac{\phi_j}{\lambda_j}\Gamma\right] \\ &\times A_j(0)^{-\frac{1-\sigma}{\varepsilon-1}} A(0)^{1-\sigma}\left((\gamma_j - 1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}\right)^{-1}. \end{aligned}$$

with $L_j(0)$ and $A_j(0)$ refering to the initial value of $L_j(t)$ and $A_j(0)$ respectively. It follows that

$$s_j(t) = \frac{L_j(0)\exp\left[\left((1-\sigma)\gamma - \frac{1-\sigma}{\varepsilon-1}g_j\right)t\right]}{\sum\limits_{k=a,m,s} L_k(0)\exp\left[\left((1-\sigma)\gamma - \frac{1-\sigma}{\varepsilon-1}g_k\right)t\right]} = \frac{L_j(0)e^{-\frac{1-\sigma}{\varepsilon-1}g_j t}}{\sum\limits_{k=a,m,s} L_k(0)e^{-\frac{1-\sigma}{\varepsilon-1}g_k t}} \tag{45}$$

Differentiating (45) with respect to t gives

$$\frac{\dot{s}_j(t)}{s_j(t)} = \frac{1-\sigma}{\varepsilon-1}\sum_{k\neq j}(g_k - g_j)\frac{L_k(0)}{L_j(0)}e^{-\frac{1-\sigma}{\varepsilon-1}g_k t}\left[\sum_k \frac{L_k(0)}{L_j(0)}e^{-\frac{1-\sigma}{\varepsilon-1}g_k t}\right]^{-2}. \tag{46}$$

Equation (46) implies that if $g_s < g_m < g_a$, $\dfrac{\dot{s}_a(t)}{s_a(t)} < 0$, $\dfrac{\dot{s}_s(t)}{s_s(t)} > 0$ while the sign of $\dfrac{\dot{s}_m(t)}{s_m(t)}$ change. To show that $s_m(t)$ exhibits a hump-shaped pattern, we will show that

$$\exists\, t^*,\ \dot{s}_m(t^*) = 0, \quad \frac{\dot{s}_m(t)}{s_m(t)} > 0, \ \ \forall\, t < t^* \quad \text{and} \quad \frac{\dot{s}_m(t)}{s_m(t)} < 0, \quad \forall\, t < t^\star.$$

According to equation (46),

$$\begin{aligned}
\frac{\dot{s}_m(t)}{s_m(t)} = 0 \quad &\Rightarrow \quad \frac{L_a(0)}{L_m(0)} e^{-\frac{1-\sigma}{\varepsilon-1} g_a t}(g_a - g_m) + \frac{L_s(0)}{L_m(0)} e^{-\frac{1-\sigma}{\varepsilon-1} g_m t}(g_s - g_m) = 0 \\
&\Rightarrow \quad e^{-\frac{1-\sigma}{\varepsilon-1}(g_a - g_s)t} = \frac{L_s(0)}{L_a(0)} \frac{g_m - g_s}{g_a - g_m} \\
&\Rightarrow \quad t^* = \frac{\varepsilon - 1}{1-\sigma} \frac{1}{g_a - g_s} \log\left[\frac{L_a(0)}{L_s(0)} \frac{g_a - g_m}{g_m - g_s}\right]
\end{aligned}$$

It's straightforward to verify that $\frac{\dot{s}_m(t)}{s_m(t)} > 0, \quad \forall\ t < t^*$ and $\frac{\dot{s}_m(t)}{s_m(t)} < 0, \ \forall\ t < t^*$. This completes the proof of Lemma 3.3.■

We formulate the following assumption to guarantee that date t^* is positive.

Assumption 3 *The economy's parameters are such that*

$$\frac{L_a(0)}{L_s(0)} \frac{g_a - g_m}{g_m - g_s} > 1.$$

where the expression of g_j and $L_j(0)$ are given in Proposition 3.1.

Definition 2 *The sectoral productivity growth gap index (SPGI) is defined as the ratio between the gap in productivity growth rates in the agriculture and industry sectors to the gap between the productivity growth rates in the industry and services sectors, which is represented by g*

$$g = \frac{g_a - g_m}{g_m - g_s}.$$

The sectoral productivity growth gap index g is positive under Assumption 3.2 and captures the tension between two opposing forces. Indeed, $g_a > g_m$ pushes labor out of agriculture to industry while $g_m > g_s$ pulls labor out of industry for services. A large *SPGI* would imply a reallocation of workers from agriculture to industry greater than the reallocation of workers from industry to the services sector, while a smaller g would imply the opposite effect. The hump-shaped in the industry sector results from these two opposing forces. At earlier stages of development when the share of agriculture is high, the flow of workers moving from agriculture to the industry sector exceeds the flow of workers moving from industry to the services sector. That corresponds to the industrialization phase. At later stages when the share of agriculture is low, the inflow of workers into industry is less than the outflow and that corresponds to the deindustrialization phase.

Having shown that the labor share in the industry sector exhibits a hump shape pattern and have also determined the date of the peak, we turn to the analyzis of the labor share in industry and the GDP at this peak.

Lemma 4 *(Labor share and GDP at the peak)*

1. *The labor share in industry at its peak is :*

$$s_m(t^*) = \left[1 + \frac{L_a(0)}{L_m(0)} \left(\frac{L_a(0)}{L_s(0)} g\right)^{-\frac{g}{1+g}} (1+g)\right]^{-1} \tag{47}$$

2. *The GDP at that peak is given by*

$$GDP(t^*) = L\, \frac{\eta_m}{\phi_m^{1-\sigma}} A_m(0)^{\frac{1-\sigma}{\varepsilon-1}} \left[1 + \frac{\eta_a}{\eta_m}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)} g\right)^{-\frac{g}{1+g}}\right. \tag{48}$$

$$\left. + \frac{\eta_s}{\eta_m}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)} g\right)^{\frac{1}{1+g}}\right]^{-\frac{1}{1-\sigma}} \left(\frac{L_a(0)}{L_s(0)}\, g\right)^{\frac{g_m}{(g_a - g_s)(1-\sigma)}}$$

with

$$\begin{aligned} L_j(0) \quad &= \quad \left[\eta_j \phi_j^{1-\sigma}\left((\gamma_j - 1)\frac{1-\sigma}{\varepsilon-1} + 1\right)\left(\frac{\varepsilon-1}{\varepsilon} L + (\gamma - r)\frac{\phi_s^\sigma}{\lambda_s \eta_s}\right) + \frac{\phi_j}{\lambda_j}\Gamma\right] \\ &\times A_j(0)^{-\frac{1-\sigma}{\varepsilon-1}} A(0)^{1-\sigma}\left((\gamma_j - 1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}\right)^{-1}. \end{aligned}$$

Proof. See Appendix A.2.

The growth rate of several variables including sectoral productivities depends on γ, the growth rate of the aggregate productivity $A(t)$. The next Lemma gives the expression of this parameter.

Lemma 5 *Along the ABGP, the growth rate of our defined aggregate productivity $A(t)$ is given by*

$$\gamma = \left((\theta - 1) + \frac{\varepsilon - 1}{\gamma_s - 1}\right)^{-1}\left(\gamma_s \frac{\lambda_s \eta_s}{\phi_s^\sigma \varepsilon}\frac{\varepsilon - 1}{\varepsilon - \gamma_s} L - \rho\right). \tag{49}$$

Proof. See Appendix A.2.

Lemma 3.5 shows us that in the long run, the economy's growth rate will depend essentially on the parameters characterizing innovation in the service sector. This result is intuitive because, along our balanced growth path, there is a perpetual reallocation of workers from other sectors to the service sector.

3.2 Premature deindustrialization

In this section, we investigate the potential drivers of premature deindustrialization. To do so, we identify the factors that would negatively influence both the labor share in industry at its peak and GDP at that peak and we identify conditions under which these factors can generate PD. Lemma 3.4 reveals that factors that jointly affect labor share in industry and the GDP at the peak are (i) the sectoral productivity growth gap index g, (ii) the relative level of initial productivity $A_a(0)/A_s(0)$ and $A_a(0)/A_m(0)$ which affect the labor share in the industry at its peak and GDP at that peak through the relative initial labor $L_a(0)/L_s(0)$, and, $L_a(0)/L_m(0)$ and; (iii) all other parameters of the model whose effect goes through the relative labor $L_a(0)/L_s(0)$.

An important result highlighted by Lemma 3.4, which constitutes one of the contributions of this paper, is that the growth rates of sectoral productivity affect the labor share in industy at its peak and GDP at that peak only through the *SGPI*. Huneeus & Rogerson (2020) found that PD can be explained by heterogeneity in agriculture productivity growth across countries. Our findings show analytically that this heterogeneity affects the labor share in industry at its peak and the GDP at that peak through our defined $SGPI$.

The following proposition shows how g, $L_a(0)/L_s(0)$, $A_a(0)/A_s(0)$, and $A_a(0)/A_m(0)$ affect the labor share in the industry at its peak and GDP at that peak.

Proposition 2 *Determinants of labor share in industry at its peak and the GDP at that peak:*

$$\begin{gathered}
\frac{\partial s_m(t^*)}{\partial g} > 0, \qquad \frac{\partial s_m(t^*)}{\partial \frac{L_a(0)}{L_s(0)}} > 0, \qquad \frac{\partial s_m(t^*)}{\partial \frac{A_a(0)}{A_m(0)}} > 0, \qquad \frac{\partial s_m(t^*)}{\partial \frac{A_a(0)}{A_s(0)}} < 0 \\
\frac{\partial GDP(t^*)}{\partial g} > 0, \qquad \frac{\partial GDP(t^*)}{\partial \frac{L_a(0)}{L_s(0)}} > 0, \qquad \frac{\partial GDP(t^*)}{\partial \frac{A_a(0)}{A_m(0)}} > 0, \qquad \frac{\partial GDP(t^*)}{\partial \frac{A_a(0)}{A_s(0)}} < 0.
\end{gathered} \tag{50}$$

Proof. See Appendix A.2.

Proposition 3.2 shows that the labor share in the industry and GDP at the peak are all increasing with the sectoral productivity growth gap index g, relative labor $L_a(0)/L_s(0)$

and relative productivity $A_a(0)/A_m(0)$ while they decrease with $A_a(0)/A_s(0)$. These results are rather intuitive. Indeed, a large g reflects the fact that the force pushing workers from agriculture into the industry sectors is relatively greater than the force pulling workers from industry into the services sectors. In addition, a high $A_a(0)/A_m(0)$ implies a higher relative price between industry and agriculture goods. This implies more workers in industry and fewer workers in agriculture according to the Baumol effect. In the same vein, a high $A_a(0)/A_s(0)$ implies a higher price of services relative to agricultural goods. It follows that a higher labor share in agriculture implies in a flow of workers moving from the agriculture to the industry sector that may exceed the flow of workers moving from industry to the services sector. Furthermore, a large value of $L_a(0)/L_s(0)$ means that there are initially more workers still in the agriculture sector, that leads to an important flow of workers moving from agriculture to industry that exceeds the flow of workers moving from industry to services.

Proposition 3.2 emphasizes that a country with a lower g, $L_a(0)/L_s(0)$ and $A_a(0)/A_m(0)$ or a higher $A_a(0)/A_s(0)$ will experience a lower labor share in the industry at its peak and GDP at that peak compared to a country with a higher g, $L_a(0)/L_s(0)$ and $A_a(0)/A_m(0)$ or a lower $A_a(0)/A_s(0)$. Hence, heterogeneity across countries in sectoral productivity growth gap index g, and relative productivities $A_a(0)/A_s(0)$, $A_a(0)/A_s(0)$ and relative initial labor $L_a(0)/L_s(0)$ can generate the heterogeneity in the level of labor share in industry at its peak and GDP at that peak. This proposition highlights that PD can result in heterogeneity across countries in relative productivities $A_a(0)/A_s(0)$ and $A_a(0)/A_s(0)$ as well as all parameters that afect g and $L_a(0)/L_s(0)$.

The question that remains to be answered is how model parameters affect the $SGPI$, g, and the relative labor $L_a(0)/L_s(0)$. Recent quantitative studies including Dennis & İşcan (2009), Uy et al. (2013) and Święcki (2017) have shown that asymmetric productivity growth across sectors is the main driver behind the structural change. Our analysis focuses on the parameters that govern sectoral productivity growth. More precisely we focus on the efficiency of R&D activity λ_j and sectoral size of innovation γ_j that can generate unequal productivity growth across sectors. Proposition 3.3 shows how these parameters affect g and $L_a(0)/L_s(0)$.

Proposition 3 *Variation of g and $L_a(0)/L_s(0)$ with efficiency of R&D activity λ_j.*

$$\frac{\partial g}{\partial \lambda_a} > 0, \qquad \frac{\partial g}{\lambda_m} < 0, \qquad \frac{\lambda g}{\partial \lambda_s} > 0. \tag{51}$$

$$\frac{\partial}{\partial \lambda_a}\left[\frac{L_a(0)}{L_s(0)}\right] < 0, \qquad \frac{\partial}{\partial \lambda_m}\left[\frac{L_a(0)}{L_s(0)}\right] = 0, \qquad \frac{\partial}{\partial \lambda_s}\left[\frac{L_a(0)}{L_s(0)}\right] > 0. \tag{52}$$

Proof. See Appendix A.2.

Proposition 3.3 says us that g increases with the efficiency of R&D activity in agriculture and services λ_a and λ_s respectively, and decreases with the efficiency of R&D activity in industry λ_m while $L_a(0)/L_s(0)$ decreases with λ_a and increases with λ_s. Therefore, a country with higher efficiency of R&D in industry and/or lower efficiency of R&D in the service sector will exhibit a lower labor share in industry at its peak and lower GDP at that peak. However, effect of efficiency of R&D activity in agriculture is not monotonic (See Appendix A.2). This parameter positively affects the labor share and GDP at the peak in the industry sector through the $SGPI$, g, and negatively affects them through $L_a(0)/L_s(0)$.

Proposition 3.3, reveals that premature deindustrialization can result from low efficiency of R&D activity in the industry sector and high efficiency of R&D activity in the service sector.

The following proposition states the implications stemming from a different size of sectoral innovation.

Proposition 4 *Variation of g and $L_a(0)/L_s(0)$ with parameters γ_j*

$$\frac{\partial g}{\partial \gamma_a} > 0, \qquad \frac{\partial g}{\partial \gamma_m} < 0, \qquad \frac{\partial g}{\partial \gamma_s} > 0, \qquad \frac{\partial}{\partial \gamma_m}\left[\frac{L_a(0)}{L_s(0)}\right] = 0 \tag{53}$$

$$sign\left(\frac{\partial}{\partial \gamma_a}\left[\frac{L_a(0)}{L_s(0)}\right]\right) = sign\left[(\varepsilon+\sigma-2)\left(\frac{\eta_a\lambda_a}{\phi_a^\sigma} - \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}\right)\right] \tag{54}$$

$$sign\left(\frac{\partial}{\partial \gamma_s}\left[\frac{L_a(0)}{L_s(0)}\right]\right) = -sign\left[(\varepsilon+\sigma-2)\left(\frac{\eta_s\lambda_s}{\phi_s^\sigma} - \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}\right)\right]. \tag{55}$$

Proof. See Appendix A.2.

Proposition 3.4 shows that g increases the size of innovation in agriculture and services

γ_a and γ_s and decreases with the size of innovation in industry γ_m. It also shows that the direction of variation of $L_a(0)/L_s(0)$ with respect to these parameters depends on the values of parameters. We assume that $\varepsilon + \sigma - 2 > 0$ because σ is positive and the lower bound of the value of elasticity of substitution across varieties ε used in literature is two[3]. Given this assumption, we will discuss different cases.

If $1 \leq \sigma\varepsilon$, the relative labor $L_a(0)/L_s(0)$ increases with γ_a and decreases with γ_s. Combining this result with the directions of variation of g with sectoral innovation sizes presented above, we can conclude that if an economy is such that $1 \leq \sigma\varepsilon$, the premature deindustrialization can result from a lower size of innovation in agriculture and a higher size of innovation in industry.

If we have rather $1 > \sigma\varepsilon$, there are several sub-cases to distinguish.
First, if the parameters are such that

$$\frac{\eta_s \lambda_s}{\phi_s^\sigma} \leq \frac{\Gamma}{E(0)} \frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2} \leq \frac{\eta_a \lambda_a}{\phi_a^\sigma},$$

g and $L_a(0)/L_s(0)$ and thereby the labor share in the industry at its peak and the GDP at that peak increase monotonously with γ_a and γ_s and decrease with γ_m. The change in respect to γ_s is not monotonic (See Appendix A.2) Therefore, the premature deindustrialization, in this case, can result from a lower size of innovation in the agriculture and services sectors and a higher size of innovation in industry.

Second, if

$$\frac{\eta_s \lambda_s}{\phi_s^\sigma}, \frac{\eta_a \lambda_a}{\phi_a^\sigma} \leq \frac{\Gamma}{E(0)} \frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2},$$

g and $L_a(0)/L_s(0)$ and thereby the labor share in the industry at its peak and the GDP at that peak increase monotonously with γ_s and decrease with γ_m while the change in respect to γ_a is not monotonic (See Appendix A.2). Therefore, premature deindustrialization can result from a lower size of innovation in the services sector and a higher size of innovation in industry sector.

[3] Acemoglu et al. (2018) for instance used $\varepsilon = 2.9$ in the model of US firm-level innovation, productivity growth, and reallocation featuring endogenous entry and exit. Uy et al. (2013) set $\varepsilon = 4$ in their Ricardian model of South Korea's structural change. Lewis et al. (2022) use $\varepsilon = 2$ in the paper where they evaluate the role of structural change on global trade. Sposi et al. (2021) also following the literature and set $\varepsilon = 2$ in the Ricardian model they use to investigate the role of mechanisms behind structural change on the explanation of deindustrialization and industry polarization in a sample of 28 countries.

Finally, if

$$\frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2} \leq \frac{\eta_s\lambda_s}{\phi_s^\sigma}, \frac{\eta_a\lambda_a}{\phi_a^\sigma},$$

g and $L_a(0)/L_s(0)$ and thereby the labor share in the industry at its peak and the GDP at that peak increase monotonously with γ_a and decrease with γ_m while the change in respect to γ_s is not monotonic (See Appendix A.2). Premature deindustrialization can result from a lower size of innovation in agriculture and a higher size of innovation in industry.

Table 3.1 summarizes our on how the sectoral efficiency of R&D activity λ_j and the sectoral size of innovation γ_j affect the labor share in industry at its peak and the GDP at that peak

Table 1: Productivity growth parameters and premature deindustrialization

	$\frac{\eta_s\lambda_s}{\phi_s^\sigma} \leq \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2} \leq \frac{\eta_a\lambda_a}{\phi_a^\sigma}$	$\frac{\eta_s\lambda_s}{\phi_s^\sigma}, \frac{\eta_a\lambda_a}{\phi_a^\sigma} \leq \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}$	$\frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2} \leq \frac{\eta_s\lambda_s}{\phi_s^\sigma}, \frac{\eta_a\lambda_a}{\phi_a^\sigma}$
$1 \leq \sigma\varepsilon$			$\frac{\partial F}{\partial\lambda_m} < 0,\ \frac{\partial F}{\partial\lambda_s} < 0$ $\frac{\partial F}{\partial\gamma_a} > 0,\ \frac{\partial F}{\partial\gamma_m} < 0$
$1 > \sigma\varepsilon$	$\frac{\partial F}{\partial\lambda_m} < 0,\ \frac{\partial F}{\partial\lambda_s} < 0$ $\frac{\partial F}{\partial\gamma_a} > 0,\ \frac{\partial F}{\partial\gamma_m} < 0\ \frac{\partial F}{\partial\gamma_s} > 0$	$\frac{\partial F}{\partial\lambda_m} < 0,\ \frac{\partial F}{\partial\lambda_s} < 0$ $\frac{\partial F}{\partial\gamma_m} < 0,\ \frac{\partial F}{\partial\gamma_s} > 0$	$\frac{\partial F}{\partial\lambda_m} < 0,\ \frac{\partial F}{\partial\lambda_s} < 0$ $\frac{\partial F}{\partial\gamma_a} > 0,\ \frac{\partial F}{\partial\gamma_m} < 0$

Note : F refers to the labor share in industry at its peak $s_m(t^*)$ and the GDP at that peak $GDP(t^*)$.

4 Conclusion

In this paper, we construct and endogenous Schumpeterian growth model of structural change to analyze the phenomenon of premature deindustrialization documented by Rodrik (2016). We show that PD can result from cross-country heterogeneity in the initial levels of productivity and in the parameters governing sectoral innovation, namely the efficiency of R&D activity and the size of innovation in each sector. We also show that this heterogeneity affects the labor share in industry at its peak and GDP at that peak through the ratio of the gap between productivity growth rates in agriculture and industry and the gap between productivity growth rates in the industry and services sectors. This ratio captures the tension between two opposing forces: the force which pushes workers from agriculture into industry and the force that pulls workers from industry into services.

Finally, it should be noted that our results depend on the assumption that the productivity growth in services is lower than that in industry and then in agriculture and on the assumption thatimposes a restriction on parameters to ensure the aggregate balanced growth path. The first hypothesis is consistent with empirical evidence of sectoral productivity from a large sample of countries.

Futur researchs can build detailed plant-level data to verify our predictions on the links between sectoral innovation parameters and premature deindustrialization that we found and in other hands use the framework develop in this model to study the opportunity to set up specific innovation subsidy policies for each sector.

References

Acemoglu, D., Akcigit, U., Alp, H., Bloom, N. & Kerr, W. (2018), 'Innovation, reallocation, and growth', *American Economic Review* **108**(11), 3450–91.

Bah, E.-h. M. (2011), 'Structural transformation paths across countries', *Emerging Markets Finance and Trade* **47**(sup2), 5–19.

Baumol, W. (1967), 'Macroeconomics of Unbalanced Growth: The Anatomy of Urban Crisis', *American Economic Review* **57**, 415–426.

Bondarev, A. & Greiner, A. (2019), 'Endogenous growth and structural change through vertical and horizontal innovations', *Macroeconomic Dynamics* **23**(1), 52–79.

Boppart, T. (2014), 'Structural Change and the Kaldor Facts in a Growth Model With Relative Price Effects and Non-Gorman Preferences', *Econometrica* **82**(6), 2167–2196.

Boppart, T. & Weiss, F. J. (2013), 'Non-homothetic preferences and industry-directed technical change'.

Buera, F. J. & Kaboski, J. P. (2009), 'Can Traditional Theories of Structural Change Fit the Data?', *Journal of the European Economic Association* **7**(2-3), 469–477.

Dennis, B. N. & İşcan, T. B. (2009), 'Engel versus Baumol: Accounting for structural change using two centuries of U.S. data', *Explorations in Economic History* **46**(2), 186–202.

Duarte, M. & Restuccia, D. (2010), 'The Role of the Structural Transformation in Aggregate Productivity', *The Quarterly Journal of Economics* **125**(1), 129–173.

Felipe, J. & Mehta, A. (2016), 'Deindustrialization? A global perspective', *Economics Letters* **149**, 148–151.

Felipe, J., Mehta, A. & Rhee, C. (2019), 'Manufacturing matters... but it's the jobs that count', *Cambridge Journal of Economics* **43**(1), 139–168.

Fujiwara, I. & Matsuyama, K. (2022), 'A technology-gap model of premature deindustrialization'.

Ghiglino, C., Nishimura, K. & Venditti, A. (2018), 'Non-Balanced Endogenous Growth and Structural Change: When Romer Meets Kaldor and Kuznets'.

Guilló, M. D., Papageorgiou, C. & Perez-Sebastian, F. (2011), 'A unified theory of structural change', *Journal of Economic Dynamics and Control* **35**(9), 1393–1404.

Haraguchi, N., Cheng, C. F. C. & Smeets, E. (2017), 'The importance of manufacturing in economic development: has this changed?', *World Development* **93**, 293–315.

Herrendorf, B., Rogerson, R. & Valentinyi, A. (2013), 'Two Perspectives on Preferences and Structural Transformation', *American Economic Review* **103**(7), 2752–2789.

Herrendorf, B., Rogerson, R. & Valentinyi, A. (2014), Chapter 6 - Growth and Structural Transformation, *in* P. Aghion & S. N. Durlauf, eds, 'Handbook of Economic Growth', Vol. 2 of *Handbook of Economic Growth*, Elsevier, pp. 855–941.

Herrendorf, B., Rogerson, R. & Valentinyi, A. (2018), Structural Change in Investment and Consumption: A Unified Approach, Working Paper, National Bureau of Economic Research.

Herrendorf, B. & Valentinyi, A. (2015), Endogenous Sector-Biased Technological Change and Industrial Policy, SSRN Scholarly Paper ID 2673621, Social Science Research Network, Rochester, NY.

Hori, T., Mizutani, N. & Uchino, T. (2018), 'Endogenous structural change, aggregate balanced growth, and optimality', *Economic Theory* **65**(1), 125–153.

Huneeus, F. & Rogerson, R. (2020), Heterogeneous paths of industrialization, Working Paper 27580, National Bureau of Economic Research.

Kongsamut, P., Rebelo, S. & Xie, D. (2001), 'Beyond Balanced Growth', *The Review of Economic Studies* **68**(4), 869–882.

Lewis, L. T., Monarch, R., Sposi, M. & Zhang, J. (2022), 'Structural change and global trade', *Journal of the European Economic Association* **20**(1), 476–512.

Moro, A. (2015), 'Structural change, growth, and volatility', *American Economic Journal: Macroeconomics* **7**(3), 259–294.

Ngai, L. R. & Pissarides, C. A. (2007), 'Structural Change in a Multisector Model of Growth', *American Economic Review* **97**(1), 429–443.

Rodrik, D. (2016), 'Premature deindustrialization', *Journal of Economic Growth* **21**(1), 1–33.

Sposi, M., Yi, K.-M. & Zhang, J. (2021), Deindustrialization and industry polarization, Working paper, National Bureau of Economic Research.

Uy, T., Yi, K.-M. & Zhang, J. (2013), 'Structural change in an open economy', *Journal of Monetary Economics* **60**(6), 667–682.

Zhang, P. (2018*a*), 'Endogenous sector-biased technical change and perpetual and transient structural change', *Journal of Economics* **123**(3), 195–223.

Zhang, W.-B. (2018*b*), 'Growth, Environmental Policy and Structural Change in a Small-Open Economy', *Politica economica* pp. 79–98.

Święcki, T. (2017), 'Determinants of structural change', *Review of Economic Dynamics* **24**, 95–131.

APPENDIX

Appendix A. Mathematics details

In this appendix, we provide some Mathematics details.

A.1: Household optimization problem

The current value Hamiltonian of household utility maximization problem is given by

$$J\left(\mathcal{A}(t), \mu(t), \{C_j(t)\}_{j=a,m,s}\right) = \frac{C(t)^{1-\theta}-1}{1-\theta} + \mu(t)\left(w(t) + r(t)\mathcal{A}(t) - \sum_{j=1}^{J} P_j(t)C_j(t)\right)$$

with

$$C(t) = \left[\sum_{j=a,m,s} \eta_j^{\frac{1}{\sigma}} C_j(t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \qquad \sum_{j=a,m,s} \eta_j = 1.$$

where $\mu(t)$ is the costate variable associated with the representative consumer's intertemporal budget constraint.

The first-order conditions in respect to $C_j(t)$ and $\mathcal{A}(t)$ are

$$\eta_j^{\frac{1}{\sigma}} C(t)^{\frac{1}{\sigma}-\theta} C_j(t)^{-\frac{1}{\sigma}} - \mu(t)P_j(t) = 0 \ \ \forall j = a, m, s \tag{A.1}$$

$$\mu(t)r(t) = -\dot{\mu}(t) + \rho\mu(t) \tag{A.2}$$

Taking the ratio of the equation (6) for two sectors i and j, we obtain :

$$\frac{P_j(t)}{P_i(t)} = \frac{\eta_j^{\frac{1}{\sigma}} C_j(t)^{-\frac{1}{\sigma}}}{\eta_i^{\frac{1}{\sigma}} C_i(t)^{-\frac{1}{\sigma}}} \tag{A.3}$$

We adopt the normalization

$$P(t) = \left[\sum_{k=1}^{J} \eta_k P_k(t)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \equiv 1.$$

Then, after some manipulation and summation, (A.3) implies

$$C(t) \quad = \quad \frac{1}{\eta_i} C_i(t) P_i(t)^{\sigma} \left[\sum_{k=1}^{J} \eta_k P_k(t)^{1-\sigma} \right]^{\frac{-\sigma}{1-\sigma}} = \frac{1}{\eta_i} P_i(t)^{\sigma} C_i(t). \tag{A.4}$$

Substituting equation (A.4) into (A.1) give

$$C_j(t) = \eta_j \mu(t)^{-\frac{1}{\theta}} P_j(t)^{-\sigma} \tag{A.5}$$

Moreover (A.4) implies that

$$P_j(t) C_i(t) = \eta_i P_j(t)^{1-\sigma} C(t) \tag{A.6}$$

Summing this (A.6) for all sector and simplifying gives (A.2) yields

$$E(t) = C(t).$$

Taking the first derivative of (A.5) with respect to time and simplifying gives (A.2) leads to the following Euler equation

$$\frac{\dot{C}_j(t)}{C_j(t)} = \frac{r(t) - \rho}{\theta} - \sigma \frac{\dot{P}_j(t)}{P_j(t)}.$$

Furthermore, differentiating equation (A.4) with respect to time t obtains

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{C}_j(t)}{C_j(t)} + \sigma \frac{\dot{P}_j(t)}{P_j(t)} = \frac{r(t) - \rho}{\theta}. \qquad \blacksquare$$

Return to model setup.

A.2: Proof of proposition 1

Simple arithmetic manipulation budget constraint in equation (3) yields

$$E(t) = w(t)L + (r - \gamma)\mathcal{A}(t). \tag{A.7}$$

Furthermore, the asset market clearing condition implies that

$$\mathcal{A}(t) = \sum_{j=a,m,s} \frac{P_j(t)}{\lambda_j A_j(t)^{\psi} A(t)^{\zeta}}. \tag{A.8}$$

Substituting equations (12) and (14) in (A.8) and according to parameters restriction in Assumption 1 we obtain

$$\mathcal{A}(t) = \sum_{j=a,m,s} \frac{\phi_j}{\lambda_j} A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}} A(t)^{2-\sigma}. \tag{A.9}$$

Combining the wage in (14) and equations (A.7) and (A.9) leads to the following expression of total expenditure

$$E(t) = A(t) \left[\frac{\varepsilon - 1}{\varepsilon} L + (r - \gamma) \sum_{j=a,m,s} \frac{\phi_j}{\lambda_j} A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}} A(t)^{1-\sigma)} \right]. \tag{A.10}$$

Moreover,

$$A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}} A(t)^{1-\sigma)} = \frac{\sum_{j=a,m,s} \frac{\phi_j}{\lambda_j} A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}}}{A(t)^{-(1-\sigma)}} = \frac{\sum_{j=a,m,s} \frac{\phi_j}{\lambda_j} A_j(0) e^{-\frac{1-\sigma}{\varepsilon-1}(g_j - g_{j_o})t}}{\sum_{j=a,m,s} \eta_j \phi_j^{1-\sigma} A_j(0) e^{-\frac{1-\sigma}{\varepsilon-1}(g_j - g_{j_o})t}},$$

where $g_j \equiv \dot{A}_j(t)/A_j(t)$ and j_o refers to the sector with the lest productivity growth rate, $j_o = \arg\min_j \{g_j,\ j = a, m, s\}$. According to this notation,

$$e^{-\frac{1-\sigma}{\varepsilon-1}(g_j - g_{j_o})t} \longrightarrow 0, \quad \forall j \neq j_o.$$

It follows that

$$A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}} A(t)^{1-\sigma)} \longrightarrow \frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o} \eta_{j_o}}.$$

Therefore, equation (A.10) can be rewritten as

$$E(t) = A(t) \left[\frac{\varepsilon - 1}{\varepsilon} L + (r - \gamma) \frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o} \eta_{j_o}} \right].$$

Given that $C_j(t) = c_j Y_j(t)$, simple arithmetic manipulation using equation (8), (12) yields

$$E(t) = \sum_{j=a,m,s} P_j(t) C_j(t) = A(t) \sum_{j=a,m,s} c_j L_j(t) \tag{A.11}$$

Furthermore, the first other condition of the utility maximization problem given in equation (6) implies that

$$\frac{C_j(t)}{C_i(t)} = \frac{\eta_j}{\eta_i} \left(\frac{P_j(t)}{P_i(t)} \right)^{-\sigma},$$

which can be rewritten as

$$\sum_{j=a,m,s} c_j L_j(t) = c_i L_i(t) \frac{\sum_{j=a,m,s} \eta_j \phi_j^{1-\sigma} A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}}}{\eta_i \phi_i^{1-\sigma} A_i(t)^{-\frac{1-\sigma}{\varepsilon-1}}}.$$

After performing simple arithmetic manipulations, we find

$$c_j L_j(t) = \frac{\eta_j \phi_j^{1-\sigma} A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}}}{A(t)^{-(1-\sigma)}} \left[\frac{\varepsilon-1}{\varepsilon} L + (r-\gamma) \frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o} \eta_{j_o}} \right].$$

It follows that

$$z_j L_j(t) = L_j(t) - \frac{\eta_j \phi_j^{1-\sigma} A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}}}{A(t)^{-(1-\sigma)}} \left[\frac{\varepsilon-1}{\varepsilon} L + (r-\gamma) \frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o} \eta_{j_o}} \right]. \tag{A.12}$$

Substituting the productivity growth rate in equation (**??**) in the arbitrary condition of R&D in (39) we arrive at

$$(2-\sigma)\gamma - r = (\gamma_j - 1) \frac{1-\sigma}{\varepsilon - 1} \lambda_j z_j Y_j(t) A_j(t)^{\psi} A(t)^{\zeta} + \left(z_j - \frac{\gamma_j}{\varepsilon} \right) Y_j(t) \lambda_j A_j(t)^{\psi} A(t)^{\zeta}.$$

After manipulations give

$$\Gamma = \left[(\gamma_j - 1) \frac{1-\sigma}{\varepsilon-1} + 1 \right] \lambda_j z_j \phi_j^{-1} A_j(t)^{\frac{1-\sigma}{\varepsilon-1}} A(t)^{\sigma-1} L_j(t) - \frac{\gamma_j}{\varepsilon} \lambda_j \phi_j^{-1} A_j(t)^{\frac{1-\sigma}{\varepsilon-1}} A(t)^{\sigma-1} L_j(t),$$

where $\Gamma \equiv (2-\sigma-\theta)\gamma - \rho$. Substituting the expression of $z_j L_j(t)$ given by the equation (A.12) in the equation below implies

$$\begin{aligned} L_j(t) \quad = \quad & \left[\eta_j \phi_j^{1-\sigma}\left((\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1\right)\left(\frac{\varepsilon-1}{\varepsilon}L+(\gamma-r)\frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o}\eta_{j_o}}\right)+\frac{\phi_j}{\lambda_j}\Gamma\right] \\ & \times A_j(t)^{-\frac{1-\sigma}{\varepsilon-1}}A(t)^{1-\sigma}\left((\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_j}{\varepsilon}\right)^{-1}. \end{aligned} \tag{A.13}$$

Finally combining equations (**??**), (A.12) and (A.13) yields the expression of productivity growth rate in sector j

$$\begin{aligned} \frac{\dot{A}_j(t)}{A_j(t)} = & \frac{\gamma_j-1}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_j}{\varepsilon}}\left[\frac{\lambda_j\eta_j}{\phi_j^{\sigma}}\left((\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1\right)\left(\frac{\varepsilon-1}{\varepsilon}L+(\gamma-r)\frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o}\eta_{j_o}}\right)+\Gamma\right] \\ & -(\gamma_j-1)\frac{\lambda_j\eta_j}{\phi_j^{\sigma}}\left(\frac{\varepsilon-1}{\varepsilon}L+(\gamma-r)\frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o}\eta_{j_o}}\right). \end{aligned} \tag{A.14}$$

To easily manipulate this expression, we denote by $\Lambda_j = (\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1$, recall also that

$$E(0) = \frac{\varepsilon-1}{\varepsilon}L+(\gamma-r)\frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o}\eta_{j_o}}.$$

Therefore

$$\begin{aligned} \frac{\dot{A}_j(t)}{A_j(t)} &= \frac{\gamma_j-1}{\Lambda_j-\frac{\gamma_j}{\varepsilon}}\left[\frac{\lambda_j\eta_j}{\phi_j^{\sigma}}\Lambda_j E(0)+\Gamma\right]-(\gamma_j-1)\frac{\lambda_j\eta_j}{\phi_j^{\sigma}}E(0) \\ &= (\gamma_j-1)\frac{\lambda_j\eta_j}{\phi_j^{\sigma}}E(0)\left[\frac{\Lambda_j}{\Lambda_j-\frac{\gamma_j}{\varepsilon}}-1\right]+\Gamma\frac{\gamma_j-1}{\Lambda_j-\frac{\gamma_j}{\varepsilon}} \\ &= \frac{\gamma_j-1}{\Lambda_j-\frac{\gamma_j}{\varepsilon}}\left[\frac{\lambda_j\eta_j}{\phi_j^{\sigma}}\frac{E(0)}{\varepsilon}\gamma_j+\Gamma\right]. \end{aligned}$$

Thus

$$\frac{\dot{A}_j(t)}{A_j(t)} = \frac{\gamma_j-1}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_j}{\varepsilon}}\left[\frac{\lambda_j\eta_j}{\phi_j^{\sigma}\varepsilon}\left(\frac{\varepsilon-1}{\varepsilon}L+(\gamma-r)\frac{\phi_{j_o}^{\sigma}}{\lambda_{j_o}\eta_{j_o}}\right)\gamma_j+\Gamma\right].$$

Return to model proposition 1.

A.2: Labor share and GDP at peak in industry

***Proof*:** Equation (45) implies that

$$\begin{aligned}
s_m(t^*)^{-1} &= \frac{L_a(0)e^{-\frac{1-\sigma}{\varepsilon-1}g_a t} + L_m(0)e^{-\frac{1-\sigma}{\varepsilon-1}g_m t^*} + L_s(0)e^{-\frac{1-\sigma}{\varepsilon-1}g_s t^*}}{L_m(0)e^{-\frac{1-\sigma}{\varepsilon-1}g_m t^*}} \\
&= \frac{L_a(0)}{L_m(0)}e^{-\frac{1-\sigma}{\varepsilon-1}(g_a-g_m)t^*} + 1 + \frac{L_s(0)}{L_m(0)}e^{-\frac{1-\sigma}{\varepsilon-1}(g_a-g_s)t^*}
\end{aligned}$$

Furthermore,

$$e^{-\frac{1-\sigma}{\varepsilon-1}(g_a-g_m)t^*} = \left(\frac{L_a(0)}{L_s(0)}\right)^{-\frac{g_a-g_m}{g_a-g_s}} g^{-\frac{g_a-g_m}{g_a-g_s}} = \left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}},$$

and

$$e^{-\frac{1-\sigma}{\varepsilon-1}(g_s-g_m)t^*} = \exp\left[-\frac{g_s-g_m}{g_a-g_s}\log\left(\frac{L_a(0)}{L_s(0)}g\right)\right] = \left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}}.$$

Thus,

$$\begin{aligned}
s_m(t^*)^{-1} &= 1 + \frac{L_a(0)}{L_m(0)}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}} + \frac{L_s(0)}{L_m(0)}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}} \\
&= 1 + \left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\left[\frac{L_a(0)}{L_m(0)} + \frac{L_s(0)}{L_m(0)}\frac{L_a(0)}{L_s(0)}g\right] \\
&= 1 + \frac{L_a(0)}{L_m(0)}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}(1+g)
\end{aligned}$$

We now return to the calculation of GDP at the peak. We start by deriving the expression of the aggregate productivity $A(t)$ at the peak. The relation (15) implies that

$$
\begin{aligned}
A(t^*)^{\sigma-1} &= \eta_a\left(\phi_a A_a(t^*)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} + \eta_m\left(\phi_m A_m(t^*)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} + \eta_s\left(\phi_s A_s(t^*)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} \\
&= \eta_a\left(\phi_a A_a(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g_a}{g_a-g_s}} + \eta_m\left(\phi_m A_m(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g_m}{g_a-g_s}} \\
&\quad + \eta_s\left(\phi_s A_s(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g_s}{g_a-g_s}} \\
&= \eta_m\left(\phi_m A_m(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g_m}{g_a-g_s}}\left[\frac{\eta_s}{\eta_m}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}}\right. \\
&\quad \left. 1+\frac{\eta_a}{\eta_m}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\right]
\end{aligned}
$$

Therefore

$$
GDP(t^*) \quad = \quad A(t^*) = LA(t^*).
$$

Return to Lemma 4

A.3: Variation of labor share and GDP at peak in industry

***Proof*:** Recall that

$$
s_m(t^*)^{-1} \quad = \quad 1 + \frac{L_a(0)}{L_m(0)}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}(1+g)
$$

Thus

$$
\frac{\partial s_m(t^*)^{-1}}{\partial g} = -\frac{L_a(0)}{L_m(0)}\left[1+\log\left(\frac{L_a(0)}{L_s(0)}g\right)\right]\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}.
$$

Under Assumption 3, $\log\left(\frac{L_a(0)}{L_s(0)}g\right) > 0$, then $\dfrac{\partial s_m(t^*)^{-1}}{\partial g} < 0$. It follows that $\dfrac{\partial s_m(t^*)}{\partial g} < 0$. Moreover,

$$
\frac{\partial s_m(t^*)^{-1}}{\partial \frac{L_a(0)}{L_s(0)}} \quad = \quad -\frac{g^2}{1+g}\frac{L_a(0)}{L_m(0)}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}-1}(1+g) < 0.
$$

We turn now to differentiation with respect to relative productivity across sectors. Recall that

$$L_j(0) = \left[E(0)\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}\right) + \frac{\phi_j}{\lambda_j}\Gamma\right]\frac{A_j(0)^{-\frac{1-\sigma}{\varepsilon-1}}}{A(0)^{\sigma-1}}\left(\frac{1-\sigma\varepsilon}{\varepsilon(\varepsilon-1)}\gamma_j + \frac{\sigma+\varepsilon-2}{\varepsilon-1}\right)^{-1}$$

Thus

$$\frac{\partial}{\partial\frac{A_j(0)}{A_i(0)}}\left[\frac{L_j(0)}{L_i(0)}\right] = -\frac{1-\sigma}{\varepsilon-1}\frac{A_i(0)}{A_j(0)}\frac{L_j(0)}{L_i(0)} < 0. \tag{A.15}$$

Using the property of the compound derivative, we obtain

$$\begin{aligned}\frac{\partial s_m(t^*)^{-1}}{\partial\frac{A_a(0)}{A_s(0)}} &= \frac{\partial s_m(t^*)^{-1}}{\partial\frac{L_a(0)}{L_s(0)}}\frac{\partial}{\partial\frac{A_a(0)}{A_s(0)}}\left[\frac{L_a(0)}{L_s(0)}\right] \\ &= g\frac{1-\sigma}{\varepsilon-1}\frac{L_a(0)}{L_m(0)}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\frac{A_s(0)}{A_a(0)} > 0.\end{aligned}$$

In some way

$$\begin{aligned}\frac{\partial s_m(t^*)^{-1}}{\partial\frac{A_a(0)}{A_m(0)}} &= \frac{\partial s_m(t^*)^{-1}}{\partial\frac{L_a(0)}{L_m(0)}}\frac{\partial}{\partial\frac{A_a(0)}{A_m(0)}}\left[\frac{L_a(0)}{L_m(0)}\right] \\ &= -\frac{1-\sigma}{\varepsilon-1}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}(1+g)\frac{A_m(0)}{A_a(0)}\frac{L_a(0)}{L_m(0)} < 0.\end{aligned}$$

Which shows that

$$\frac{\partial s_m(t^*)}{\partial\frac{A_a(0)}{A_s(0)}} < 0, \quad \text{and} \quad \frac{\partial s_m(t^*)}{\partial\frac{A_a(0)}{A_m(0)}} > 0.$$

We turn to the variation of the GDP at the peak. We first recall that

$$GDP(t^*) = LA(t^*),$$

where $A(t^*)$ is given by

$$
\begin{aligned}
A(t^*)^{\sigma-1} &= \eta_a \left(\phi_a A_a(t^*)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} + \eta_m \left(\phi_m A_m(t^*)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} + \eta_s \left(\phi_s A_s(t^*)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} \\
&= \eta_a \left(\phi_a A_a(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} \left(\frac{L_a(0)}{L_s(0)}\, g\right)^{-\frac{g_a}{g_a-g_s}} + \eta_m \left(\phi_m A_m(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} \left(\frac{L_a(0)}{L_s(0)}\, g\right)^{-\frac{g_m}{g_a-g_s}} \\
&\quad + \eta_s \left(\phi_s A_s(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} \left(\frac{L_a(0)}{L_s(0)}\, g\right)^{-\frac{g_s}{g_a-g_s}} \\
&= \eta_m \left(\phi_m A_m(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma} \left(\frac{L_a(0)}{L_s(0)}\, g\right)^{-\frac{g_m}{g_a-g_s}} \left[\frac{\eta_s}{\eta_m}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma} \left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \left(\frac{L_a(0)}{L_s(0)} g\right)^{\frac{1}{1+g}}\right. \\
&\quad \left. 1 + \frac{\eta_a}{\eta_m}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma} \left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \left(\frac{L_a(0)}{L_s(0)} g\right)^{-\frac{g}{1+g}}\right]
\end{aligned}
$$

We will focus on the derivation of $A(t^*)$ to analyze the variation $GDP(t^*)$. To simplify our calculations, we do the intermediate differentiation.

$$
\frac{\partial A(t^*)^{\sigma-1}}{\partial \frac{L_a(0)}{L_s(0)}} = -\sum_{j=a,m,s} \frac{g_j}{g(g_a-g_s)} \eta_j \phi_j^{1-\sigma} A_j(0)^{-\frac{1-\sigma}{\varepsilon-1}} \left(\frac{L_a(0)}{L_s(0)}\, g\right)^{-\frac{g_j}{g_a-g_s}-1} < 0.
$$

Furthermore,

$$
\begin{aligned}
\frac{\partial}{\partial g}\left[\left(\frac{L_a(0)}{L_s(0)}\, g\right)^{-\frac{g_m}{g_a-g_s}}\right] &= -\frac{g_m}{g_a-g_s}\frac{1}{g}\left(\frac{L_a(0)}{L_s(0)} g\right)^{-\frac{g_m}{g_a-g_s}} \\
\frac{\partial}{\partial g}\left[\left(\frac{L_a(0)}{L_s(0)} g\right)^{-\frac{g}{1+g}}\right] &= -\frac{1}{1+g}\left[\frac{1}{1+g}\log\left(\frac{L_a(0)}{L_s(0)} g\right) + 1\right]\left(\frac{L_a(0)}{L_s(0)} g\right)^{-\frac{g}{1+g}} \\
\frac{\partial}{\partial g}\left[\left(\frac{L_a(0)}{L_s(0)} g\right)^{\frac{1}{1+g}}\right] &= -\frac{1}{1+g}\left[\frac{1}{1+g}\log\left(\frac{L_a(0)}{L_s(0)} g\right) - \frac{1}{g}\right]\left(\frac{L_a(0)}{L_s(0)} g\right)^{\frac{1}{1+g}}.
\end{aligned}
$$

Given these intermediate calculations, we can write

$$
\begin{aligned}
\frac{\partial A(t^*)^{\sigma-1}}{\partial g} &= -F_1 \frac{g_m}{g_a - g_s}\left[1+\frac{\eta_a}{\eta_m}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\right. \\
&\left.+\frac{\eta_s}{\eta_m}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}}\right] \\
&- F_1\frac{g}{1+g}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\left\{\frac{\eta_a}{\eta_m}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left[\frac{1}{1+g}\log\left(\frac{L_a(0)}{L_s(0)}g\right)+1\right]\right. \\
&\left.+\quad\frac{\eta_s}{\eta_m}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left[\frac{1}{1+g}\log\left(\frac{L_a(0)}{L_s(0)}g\right)-\frac{1}{g}\right]\frac{L_a(0)}{L_s(0)}g\right\}
\end{aligned}
$$

where

$$
F_1 = \left(\phi_m A_m(0)^{-\frac{1}{\varepsilon-1}}\right)^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g_m}{g_a-g_s}}.
$$

Performing some arithmetical manipulation yields

$$
\begin{aligned}
\frac{\partial A(t^*)^{\sigma-1}}{\partial g} &= -F_1\eta_m \frac{g_m}{g_a - g_s}\frac{1}{g}\left[1+\frac{\eta_a}{\eta_m}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\right] \\
&- F_1\frac{\eta_s}{g}\frac{g_m}{g_a-g_s}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}} \\
&- F_1\frac{\eta_a}{1+g}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left[\frac{1}{1+g}\log\left(\frac{L_a(0)}{L_s(0)}g\right)+1\right]\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}} \\
&- F_1\frac{\eta_s}{1+g}\frac{1}{1+g}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\log\left(\frac{L_a(0)}{L_s(0)}g\right)\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}} \\
&+ F_1\frac{\eta_s}{g}\frac{1}{1+g}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}}
\end{aligned}
$$

It follows that

$$\frac{\partial A(t^*)^{\sigma-1}}{\partial g} = -F_1\eta_m \frac{g_m}{g_a - g_s}\frac{1}{g}\left[1 + \frac{\eta_a}{\eta_m}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\right]$$

$$- F_1\frac{\eta_s}{g}\frac{g_s}{g_a - g_s}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}}$$

$$- F_1\frac{\eta_a}{1+g}\left(\frac{\phi_a}{\phi_m}\right)^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left[\frac{1}{1+g}\log\left(\frac{L_a(0)}{L_s(0)}g\right) + 1\right]\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}$$

$$- F_1\frac{\eta_s}{1+g}\frac{1}{1+g}\left(\frac{\phi_s}{\phi_m}\right)^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\log\left(\frac{L_a(0)}{L_s(0)}g\right)\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}}$$

All the lines of this differentiation are less than zero. Thus $\dfrac{\partial A(t^*)^{\sigma-1}}{\partial g} < 0$. We conclude that $\dfrac{\partial A(t^*)}{\partial g} > 0$ and then $\dfrac{\partial GDP(t^*)}{\partial g} > 0$.

We now compute derive GDP with respect to the relative level of initial productivity.

$$A(t^*)^{\sigma-1} = A_s(0)^{-\frac{1-\sigma}{\varepsilon-1}}\Bigg[\eta_s\phi_s^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{\frac{1}{1+g}} + \eta_a\phi_a^{1-\sigma}\left(\frac{A_a(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g_a}{g_a-g_s}}$$

$$+\eta_m\phi_m^{1-\sigma}\left(\frac{A_m(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g}{1+g}}\Bigg]$$

$$\frac{1}{A_s(0)^{-\frac{1-\sigma}{\varepsilon-1}}}\frac{\partial A(t^*)^{\sigma-1}}{\partial\frac{A_a(0)}{A_s(0)}} = \frac{g}{1+g}\eta_s\phi_s^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{\frac{1}{1+g}-1}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial\frac{A_a(0)}{A_s(0)}}$$

$$-\frac{1-\sigma}{\varepsilon-1}\eta_a\phi_a^{1-\sigma}\left(\frac{A_a(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}-1}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g_a}{g_a-g_s}}$$

$$-\frac{g_a}{g_a-g_s}\eta_a g\phi_a^{1-\sigma}\left(\frac{A_a(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g_a}{g_a-g_s}-1}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial\frac{A_a(0)}{A_s(0)}}$$

$$-\frac{g}{1+g}g\eta_m\phi_m^{1-\sigma}\left(\frac{A_m(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\frac{g}{1+g}-1}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial\frac{A_a(0)}{A_s(0)}}$$

Substituting equation (A.15) in the third row gives

$$\frac{1}{A_s(0)^{-\frac{1-\sigma}{\varepsilon-1}}}\frac{\partial A(t^*)^{\sigma-1}}{\partial\frac{A_a(0)}{A_s(0)}} = \frac{g}{1+g}\eta_s\phi_s^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}-1}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial\frac{A_a(0)}{A_s(0)}}$$
$$-\frac{1-\sigma}{\varepsilon-1}\eta_a\phi_a^{1-\sigma}\left(\frac{A_a(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}-1}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g_a}{g_a-g_s}}$$
$$\frac{g_a}{g_a-g_s}\eta_a\phi_a^{1-\sigma}\left(\frac{A_a(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}-1}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g_a}{g_a-g_s}}\frac{1-\sigma}{\varepsilon-1}$$
$$-\frac{g}{1+g}g\eta_m\phi_m^{1-\sigma}\left(\frac{A_m(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}-1}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial\frac{A_a(0)}{A_s(0)}}$$

Therefore

$$\frac{1}{A_s(0)^{-\frac{1-\sigma}{\varepsilon-1}}}\frac{\partial A(t^*)^{\sigma-1}}{\partial\frac{A_a(0)}{A_s(0)}} = \frac{g}{1+g}\eta_s\phi_s^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}-1}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial\frac{A_a(0)}{A_s(0)}}$$
$$\frac{1-\sigma}{\varepsilon-1}\frac{g_s}{g_a-g_s}\eta_a\phi_a^{1-\sigma}\left(\frac{A_a(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}-1}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g_a}{g_a-g_s}}$$
$$-\frac{g}{1+g}g\eta_m\phi_m^{1-\sigma}\left(\frac{A_m(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}-1}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial\frac{A_a(0)}{A_s(0)}}$$

Since $\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial\frac{A_a(0)}{A_s(0)}} < 0$, the equations above imply that $\frac{\partial A(t^*)^{\sigma-1}}{\partial\frac{A_a(0)}{A_s(0)}} > 0$. Thus $\frac{\partial GDP(t^*)}{\partial\frac{A_a(0)}{A_s(0)}} < 0$.

Finally,

$$A(t^*)^{\sigma-1} = A_m(0)^{-\frac{1-\sigma}{\varepsilon-1}}\left[\eta_m\phi_m^{1-\sigma}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{-g}{1+g}} + \eta_a\phi_a^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g_a}{g_a-g_s}}\right.$$
$$\left.+\eta_s\phi_s^{1-\sigma}\left(\frac{A_s(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}}\left(\frac{L_a(0)}{L_s(0)}g\right)^{\frac{1}{1+g}}\right]$$

$$\frac{\partial A(t^*)^{\sigma-1}}{\partial\frac{A_a(0)}{A_m(0)}} = -\frac{1-\sigma}{\varepsilon-1}A_m(0)^{-\frac{1-\sigma}{\varepsilon-1}}\eta_a\phi_a^{1-\sigma}\left(\frac{A_a(0)}{A_m(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}-1}\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g_a}{g_a-g_s}} < 0.$$

It follows that $\frac{\partial GDP(t^*)}{\partial\frac{A_a(0)}{A_m(0)}} > 0$. Return to proposition 2.

A.3: Static comparative in g and $L_a(0)/L_s(0)$

Proof Recall that

$$g_j = \frac{\dot{A}_j(t)}{A_j(t)} = \frac{\gamma_j - 1}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}} \left[\frac{\lambda_j\eta_j}{\phi_j^\sigma \varepsilon} \left(\frac{\varepsilon-1}{\varepsilon} L + (\gamma - r)\frac{\phi_{j0}^\sigma}{\lambda_{j0}\eta_{j0}} \right) \gamma_j + \Gamma \right]$$

$$= \frac{\gamma_j - 1}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}} \left[\frac{\lambda_j\eta_j}{\phi_j^\sigma \varepsilon} E(0)\gamma_j + \Gamma \right] \tag{A.16}$$

Thus, differentiating (A.16) with respect to γ_j

$$\frac{\partial g_j}{\partial \gamma_j} = \frac{\frac{\varepsilon}{\varepsilon-1}}{\left[(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}\right]^2} \left(\frac{\lambda_j\eta_j}{\phi_j^\sigma \varepsilon} E(0)\gamma_j + \Gamma \right) + \frac{\lambda_j\eta_j}{\phi_j^\sigma \varepsilon} E(0) \frac{\gamma_j - 1}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}} > 0.$$

Differentiating (A.16) with respect to λ_j for $j \neq j_o$,

$$\frac{\partial g_j}{\partial \lambda_j} = \frac{\gamma_j - 1}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}} \frac{\eta_j}{\phi_j^\sigma \varepsilon} E(0)\gamma_j > 0$$

Since $E(0) = \frac{\lambda_j\eta_j}{\phi_j^\sigma \varepsilon} \left(\frac{\varepsilon-1}{\varepsilon} L + (\gamma - r)\frac{\phi_{j0}^\sigma}{\lambda_{j0}\eta_{j0}} \right)$ depend on λ_{j_o} and given that under Assumption 2, $j_o = s$, we can have, differentiating (A.16) with respect to λ_s for $j \neq j_o$ gives

$$\frac{\partial g_j}{\partial \lambda_s} = -\frac{\gamma_j - 1}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}} \frac{\lambda_j\eta_j}{\phi_j^\sigma \varepsilon} (\gamma - r) \frac{\phi_s^\sigma}{\lambda_s^2 \eta_s} \gamma_j < 0.$$

Moreover,

$$\frac{\partial g_s}{\partial \lambda_s} = \frac{\gamma_s - 1}{(\gamma_s-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_s}{\varepsilon}} \frac{\eta_s}{\phi_s^\sigma \varepsilon} \frac{\varepsilon-1}{\varepsilon} L \left((\gamma_s - 1)\frac{1-\sigma}{\varepsilon-1} + 1 \right) > 0.$$

We use these intermediate differentiation to perform the differentiation of g with respect to γ_j and λ_j as follows.

$$\frac{\partial g}{\partial \gamma_a} = \frac{1}{g_m - g_s}\frac{\partial g_a}{\partial \gamma_a} > 0, \quad \frac{\partial g}{\partial \gamma_m} = -\frac{g_a - g_s}{(g_m - g_s)^2}\frac{\partial g_m}{\partial \gamma_m} < 0, \quad \frac{\partial g}{\partial \gamma_s} = \frac{g_a - g_m}{(g_m - g_s)^2}\frac{\partial g_s}{\partial \gamma_s} > 0. \tag{A.17}$$

$$\frac{\partial g}{\partial \lambda_a} = \frac{1}{g_m - g_s}\frac{\partial g_a}{\partial \lambda_a} > 0, \quad \frac{\partial g}{\partial \lambda_m} = -\frac{g_a - g_s}{(g_m - g_s)^2}\frac{\partial g_m}{\partial \gamma_m} < 0, \quad \frac{\partial g}{\partial \lambda_s} = \frac{g_a - g_m}{(g_m - g_s)^2}\frac{\partial g_s}{\partial \lambda_s} > 0.$$

Statistic comparative in $L_a(0)/L_s(0)$ Recall that

$$\frac{L_j(0)}{L_i(0)} = \left(\frac{A_j(0)}{A_i(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \frac{E(0)\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}\right) + \frac{\phi_j}{\lambda_j}\Gamma}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}} \frac{(\gamma_i-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_i}{\varepsilon}}{E(0)\eta_i\phi_i^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}\right) + \frac{\phi_i}{\lambda_i}\Gamma}$$

$$= \left(\frac{A_j(0)}{A_i(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \frac{E(0)\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}\right) + \frac{\phi_j}{\lambda_j}\Gamma}{\frac{1-\sigma\varepsilon}{\varepsilon(\varepsilon-1)}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}} \frac{\frac{1-\sigma\varepsilon}{\varepsilon(\varepsilon-1)}\gamma_i + \frac{\varepsilon+\sigma-2}{\varepsilon-1}}{E(0)\eta_i\phi_i^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}\right) + \frac{\phi_i}{\lambda_i}\Gamma}$$

Thus,

$$\frac{\partial}{\partial\gamma_j}\left[\frac{L_j(0)}{L_i(0)}\right] = F_2 \frac{E(0)\eta_j\phi_j^{1-\sigma}\frac{1-\sigma}{\varepsilon-1}\cdot\frac{\varepsilon+\sigma-2}{\varepsilon-1} - \frac{1-\sigma\varepsilon}{\varepsilon(\varepsilon-1)}\cdot\left(E(0)\eta_j\phi_j^{1-\sigma}\frac{\varepsilon+\sigma-2}{\varepsilon-1} + \frac{\phi_j\Gamma}{\lambda_j}\right)}{\left[(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}\right]^2}$$

$$= F_2 \frac{E(0)\eta_j\phi_j^{1-\sigma}\frac{\varepsilon+\sigma-2}{\varepsilon-1}\cdot\frac{\varepsilon-\sigma\varepsilon-1+\sigma\varepsilon}{\varepsilon(\varepsilon-1)} - \frac{1-\sigma\varepsilon}{\varepsilon(\varepsilon-1)}\frac{\phi_j\Gamma}{\lambda_j}}{\left[(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}\right]^2}$$

$$= \frac{\frac{F_2E(0)}{\varepsilon(\varepsilon-1)}\frac{\phi_j}{\lambda_j}}{\left[(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}\right]^2}(\varepsilon+\sigma-2)\left(\frac{\eta_j\lambda_j}{\phi_j^\sigma} - \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}\right)$$

where

$$F_2 = \left(\frac{A_j(0)}{A_i(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \frac{(\gamma_i-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_i}{\varepsilon}}{E(0)\eta_i\phi_i^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}\right) + \frac{\phi_i}{\lambda_i}\Gamma}$$

Thus

$$\text{sign}\left(\frac{\partial}{\partial\gamma_a}\left[\frac{L_a(0)}{L_s(0)}\right]\right) = \text{sign}\left[(\varepsilon+\sigma-2)\left(\frac{\eta_a\lambda_a}{\phi_a^\sigma} - \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}\right)\right]$$

$$\text{sign}\left(\frac{\partial}{\partial\gamma_a}\left[\frac{L_a(0)}{L_m(0)}\right]\right) = \text{sign}\left[(\varepsilon+\sigma-2)\left(\frac{\eta_a\lambda_a}{\phi_a^\sigma} - \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}\right)\right]$$

Moreover,

$$\frac{\partial}{\partial\gamma_i}\left[\frac{L_j(0)}{L_i(0)}\right] = F_3 \frac{\frac{1-\sigma\varepsilon}{\varepsilon(\varepsilon-1)}\left(\frac{\varepsilon+\sigma-2}{\varepsilon-1}E(0)\eta_i\phi_i^{1-\sigma} + \frac{\phi_i\Gamma}{\lambda_i}\right) - \frac{\varepsilon+\sigma-2}{\varepsilon-1}\cdot\frac{1-\sigma}{\varepsilon-1}E(0)\eta_i\phi_i^{1-\sigma}}{\left[E(0)\eta_i\phi_i^{1-\sigma}\left((\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}\right) + \frac{\phi_i\Gamma}{\lambda_i}\right]^2}$$

$$= -\frac{\frac{F_3E(0)}{\varepsilon(\varepsilon-1)}\frac{\phi_i}{\lambda_i}(\varepsilon+\sigma-2)}{\left[E(0)\eta_i\phi_i^{1-\sigma}\left((\gamma_j-1)\frac{1-\sigma}{\varepsilon-1} + 1 - \frac{\gamma_j}{\varepsilon}\right) + \frac{\phi_i\Gamma}{\lambda_i}\right]^2}\left(\frac{\eta_i\lambda_i}{\phi_i^\sigma} - \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}\right)$$

$$F_3 = \left(\frac{A_j(0)}{A_i(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \frac{E(0)\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}\right) + \frac{\phi_j}{\lambda_j}\Gamma}{\frac{1-\sigma\varepsilon}{\varepsilon(\varepsilon-1)}\gamma_j + \frac{\varepsilon+\sigma-2}{\varepsilon-1}}$$

Thus

$$\text{sign}\left(\frac{\partial}{\partial\gamma_s}\left[\frac{L_a(0)}{L_s(0)}\right]\right) = -\text{sign}\left[(\varepsilon+\sigma-2)\left(\frac{\eta_s\lambda_s}{\phi_s^\sigma} - \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}\right)\right].$$

$$\text{sign}\left(\frac{\partial}{\partial\gamma_m}\left[\frac{L_a(0)}{L_m(0)}\right]\right) = -\text{sign}\left[(\varepsilon+\sigma-2)\left(\frac{\eta_m\lambda_m}{\phi_m^\sigma} - \frac{\Gamma}{E(0)}\frac{1-\sigma\varepsilon}{\varepsilon+\sigma-2}\right)\right].$$

Furthermore

$$\frac{\partial}{\partial\gamma_m}\left[\frac{L_a(0)}{L_s(0)}\right] = 0, \quad \text{and} \quad \frac{\partial}{\partial\gamma_s}\left[\frac{L_a(0)}{L_m(0)}\right] = 0.$$

We turn to the differentiation with respect to xxx

$$\frac{\partial}{\partial\lambda_j}\left[\frac{L_j(0)}{L_i(0)}\right] = \frac{-\Gamma\frac{\phi_j}{\lambda_j^2}}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_j}{\varepsilon}}\left(\frac{A_j(0)}{A_i(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \frac{(\gamma_i-1)\frac{1-\sigma}{\varepsilon-1}+1}{E(0)\eta_i\phi_i^{1-\sigma}\left((\gamma_i-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_i}{\varepsilon}\right)+\frac{\phi_i}{\lambda_i}\Gamma}$$

$$\begin{aligned}\frac{\partial}{\partial\lambda_i}\left[\frac{L_j(0)}{L_i(0)}\right] = &\frac{\Gamma\frac{\phi_i}{\lambda_j^2}\left((\gamma_i-1)\frac{1-\sigma}{\varepsilon-1}+1\right)}{\left[E(0)\eta_i\phi_i^{1-\sigma}\left((\gamma_i-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_i}{\varepsilon}\right)+\frac{\phi_i}{\lambda_i}\Gamma\right]^2}\left(\frac{A_j(0)}{A_i(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \\ &\times\frac{E(0)\eta_j\phi_j^{1-\sigma}\left((\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_j}{\varepsilon}\right)+\frac{\phi_i}{\lambda_j}\Gamma}{(\gamma_i-1)\frac{1-\sigma}{\varepsilon-1}+1}\end{aligned}$$

Given the following calculation, we can write

$$\frac{\partial}{\partial\lambda_a}\left[\frac{L_a(0)}{L_s(0)}\right] < 0, \quad \frac{\partial}{\partial\lambda_a}\left[\frac{L_a(0)}{L_m(0)}\right] < 0, \quad \frac{\partial}{\partial\lambda_m}\left[\frac{L_a(0)}{L_s(0)}\right] = 0, \quad \frac{\partial}{\partial\lambda_m}\left[\frac{L_a(0)}{L_m(0)}\right] > 0.$$

We compute no differentiation with respect to λ_s. Recall that

$$E(0) = \frac{\varepsilon-1}{\varepsilon}L + (\gamma - r)\frac{\phi_s^\sigma}{\lambda_s\eta_s} \implies \frac{\partial E(0)}{\lambda_s} = -(\gamma-r)\frac{\phi_s^\sigma}{\lambda_s^2\eta_s}$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial \lambda_s}\left[\frac{L_j(0)}{L_s(0)}\right] \propto & -(\gamma-r)\frac{\phi_s^{\sigma}}{\lambda_s^2\eta_s}\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\\
&\times\left[E(0)\eta_s\phi_s^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_s+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)+\frac{\phi_s\Gamma}{\lambda_s}\right]\\
&-\left[-(\gamma-r)\frac{\phi_s^{\sigma}}{\lambda_s^2\eta_s}\eta_s\phi_s^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_s+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)-\frac{\phi_s\Gamma}{\lambda_s^2}\right]\\
&\times\left[E(0)\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)+\frac{\phi_j\Gamma}{\lambda_j}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial \lambda_s}\left[\frac{L_j(0)}{L_s(0)}\right] \propto & -(\gamma-r)E(0)\eta_s\phi_s^{1-\sigma}\frac{\phi_s^{\sigma}}{\lambda_s^2\eta_s}\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\left(\frac{1-\sigma}{\varepsilon-1}\gamma_s+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\\
&-(\gamma-r)\frac{\phi_s\Gamma}{\lambda_s}\frac{\phi_s^{\sigma}}{\lambda_s^2\eta_s}\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\\
&+(\gamma-r)E(0)\eta_j\phi_j^{1-\sigma}\frac{\phi_s^{\sigma}}{\lambda_s^2\eta_s}\eta_s\phi_s^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_s+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\\
&+(\gamma-r)\frac{\phi_j\Gamma}{\lambda_j}\frac{\phi_s^{\sigma}}{\lambda_s^2\eta_s}\eta_s\phi_s^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_s+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\\
&+\frac{\phi_s\Gamma}{\lambda_s^2}E(0)\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\\
&+\frac{\phi_s\Gamma}{\lambda_s^2}\frac{\phi_j\Gamma}{\lambda_j}\end{aligned}$$

Substituting the expression of $E(0)$ and rearranging yields

$$\begin{aligned}\frac{\partial}{\partial \lambda_s}\left[\frac{L_j(0)}{L_s(0)}\right] \propto & -(\gamma-r)\frac{\phi_s\Gamma}{\lambda_s}\frac{\phi_s^{\sigma}}{\lambda_s^2\eta_s}\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\\
&+(\gamma-r)\frac{\phi_j\Gamma}{\lambda_j}\frac{\phi_s^{\sigma}}{\lambda_s^2\eta_s}\eta_s\phi_s^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_s+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)+\frac{\phi_s\Gamma}{\lambda_s^2}\frac{\phi_j\Gamma}{\lambda_j}\\
&+\frac{\phi_s\Gamma}{\lambda_s^2}\frac{\varepsilon-1}{\varepsilon}L\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\\
&+\frac{\phi_s\Gamma}{\lambda_s^2}(\gamma-r)\frac{\phi_s^{\sigma}}{\lambda_s\eta_s}\eta_j\phi_j^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\end{aligned}$$

It follows that

$$\frac{\partial\frac{L_j(0)}{L_s(0)}}{\partial\lambda_s}=F_5\left[(\gamma-r)\left(\frac{1-\sigma}{\varepsilon-1}\gamma_s+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)+\Gamma+\frac{\varepsilon-1}{\varepsilon}L\frac{\eta_j\lambda_j}{\phi_j^{\sigma}}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_j+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)\right] \tag{A.18}$$

where

$$F_5 = \left(\frac{A_j(0)}{A_s(0)}\right)^{-\frac{1-\sigma}{\varepsilon-1}} \frac{(\gamma_s-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_s}{\varepsilon}}{(\gamma_j-1)\frac{1-\sigma}{\varepsilon-1}+1-\frac{\gamma_j}{\varepsilon}} \left(E(0)\eta_s\phi_s^{1-\sigma}\left(\frac{1-\sigma}{\varepsilon-1}\gamma_s+\frac{\varepsilon+\sigma-2}{\varepsilon-1}\right)+\frac{\phi_s}{\lambda_s}\Gamma\right)^{-2}\Gamma\frac{\phi_j}{\lambda_j}\frac{\phi_s}{\lambda_s^2}$$

Therefore, $\dfrac{\partial}{\partial\lambda_s}\left[\dfrac{L_a(0)}{L_s(0)}\right] > 0$. Return to proposition 3.

A.4: Variation of $L_a(0)/L_s(0)$

$$\frac{\partial}{\partial X}\left[\frac{L_a(0)}{L_m(0)}(1+g)\right] = \frac{\partial\frac{L_a(0)}{L_m(0)}}{\partial X}(1+g)+\frac{L_a(0)}{L_m(0)}\frac{\partial g}{\partial X}$$

$$\begin{aligned}
\frac{\partial}{\partial X}\left[\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\right] &= \frac{\partial}{\partial X}\exp\left\{-\frac{g}{1+g}\left[\log\left(\frac{L_a(0)}{L_s(0)}\right)+\log g\right]\right\} \\
&= \left[-\frac{1}{(1+g)^2}\log\left(\frac{L_a(0)}{L_s(0)}g\right)-\frac{g}{1+g}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial X}\frac{L_s(0)}{L_a(0)}-\frac{1}{1+g}\frac{\partial g}{\partial X}\right]\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial s_m^{-1}(t^*)}{\partial X} &= \left[\frac{\partial\frac{L_a(0)}{L_m(0)}}{\partial X}(1+g)+\frac{L_a(0)}{L_m(0)}\frac{\partial g}{\partial X}\right]\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}+\frac{L_a(0)}{L_m(0)}(1+g)\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}} \\
&\quad\times\left[-\frac{1}{(1+g)^2}\log\left(\frac{L_a(0)}{L_s(0)}g\right)-\frac{g}{1+g}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial X}\frac{L_s(0)}{L_a(0)}-\frac{1}{1+g}\frac{\partial g}{\partial X}\right] \\
&= \frac{L_a(0)}{L_m(0)}(1+g)\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}\left[\frac{\partial\frac{L_a(0)}{L_m(0)}}{\partial X}\frac{L_m(0)}{L_a(0)}+\frac{1}{(1+g)}\frac{\partial g}{\partial X}\right. \\
&\quad\left.-\frac{1}{(1+g)^2}\log\left(\frac{L_a(0)}{L_s(0)}g\right)-\frac{g}{1+g}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial X}\frac{L_s(0)}{L_a(0)}-\frac{1}{1+g}\frac{\partial g}{\partial X}\right]
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial s_m^{-1}(t^*)}{\partial X} &= \left[\frac{\partial\frac{L_a(0)}{L_m(0)}}{\partial X}\frac{L_m(0)}{L_a(0)}-\frac{1}{(1+g)^2}\log\left(\frac{L_a(0)}{L_s(0)}g\right)-\frac{g}{1+g}\frac{\partial\frac{L_a(0)}{L_s(0)}}{\partial X}\frac{L_s(0)}{L_a(0)}\right] \\
&\quad\times\frac{L_a(0)}{L_m(0)}(1+g)\left(\frac{L_a(0)}{L_s(0)}g\right)^{-\frac{g}{1+g}}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial \gamma_j}\left[\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\chi_i}\right] &= \frac{\partial}{\partial \gamma_j}\exp\left(-\chi_i \log\frac{L_a(0)}{L_s(0)} - \chi \log g\right) \\
&= \left[-\frac{\partial \chi_i}{\partial \gamma_j}\log\left(\frac{L_a(0)}{L_s(0)}\right) - \chi_i \frac{\partial \frac{L_a(0)}{L_s(0)}}{\partial \gamma_j}\frac{L_s(0)}{L_a(0)} - \frac{\partial \chi_i}{\partial \gamma_j}\log g - \frac{\chi_i}{g}\frac{\partial g}{\partial \gamma_j}\right]\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\chi_i} \\
&= \left[-\frac{\partial \chi_i}{\partial \gamma_j}\log\left(\frac{L_a(0)}{L_s(0)}g\right) - \chi_i \frac{\partial \frac{L_a(0)}{L_s(0)}}{\partial \gamma_j}\frac{L_s(0)}{L_a(0)} - \frac{\chi_i}{g}\frac{\partial g}{\partial \gamma_j}\right]\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\chi_i}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A(t^*)^{\sigma-1}}{\partial \gamma_j} &= \sum_{i=a,m,s}\eta_i \phi_i^{1-\sigma} A_i(0)^{-\frac{1-\sigma}{\varepsilon-1}}\left[-\frac{\partial \chi_i}{\partial \gamma_j}\log\left(\frac{L_a(0)}{L_s(0)}g\right) - \chi_i \frac{\partial \frac{L_a(0)}{L_s(0)}}{\partial \gamma_j}\frac{L_s(0)}{L_a(0)} - \frac{\chi_i}{g}\frac{\partial g}{\partial \gamma_j}\right]\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\chi_i} \\
&= -\log\left(\frac{L_a(0)}{L_s(0)}g\right)\sum_{i=a,m,s}\eta_i \phi_i^{1-\sigma} A_i(0)^{\frac{1-\sigma}{\varepsilon-1}}\frac{\partial \chi_i}{\partial \gamma_j}\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\chi_i} \\
&\quad - \left(\frac{\partial \frac{L_a(0)}{L_s(0)}}{\partial \gamma_j}\frac{L_s(0)}{L_a(0)} + \frac{1}{g}\frac{\partial g}{\partial \gamma_j}\right)\sum_{i=a,m,s}\eta_i \phi_i^{1-\sigma} A_i(0)^{-\frac{1-\sigma}{\varepsilon-1}}\chi_i\left(\frac{L_a(0)}{L_s(0)}\,g\right)^{-\chi_i}
\end{aligned}
$$

Return to proposition 3.